Worksheet 23, Math H53
Curl and Divergence

Tuesday, April 21, 2013

1. Find the curl and the divergence of the vector field $\mathbf{F}(x, y, z) = xye^z\mathbf{i} + yze^x\mathbf{k}$.

2. Find the curl and the divergence of the vector field $\mathbf{F}(x, y, z) = \langle x/y, y/z, z/x \rangle$.

3. Let f be a scalar field and $\mathbf{F}$ a vector field. Determine whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.
   (a) $\operatorname{curl} f$
   (b) ∇f
   (c) $\operatorname{div} \mathbf{F}$
   (d) $\operatorname{curl}(\nabla f)$
   (e) $\nabla \mathbf{F}$
   (f) $\nabla(\operatorname{div} \mathbf{F})$
   (g) $\operatorname{div}(\nabla f)$
   (h) $\nabla(\operatorname{div} f)$
   (i) $\operatorname{curl}(\operatorname{curl} \mathbf{F})$
   (j) $\operatorname{div}(\operatorname{div} \mathbf{F})$
   (k) $(\nabla f) \times (\operatorname{div} \mathbf{F})$
   (l) $\operatorname{div}(\operatorname{curl}(\nabla f))$

4. Determine whether or not the vector field $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$ is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$.

5. Determine whether or not the vector field $\mathbf{F}(x, y, z) = xyz^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$ is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$.

6. Determine whether or not the vector field $\mathbf{F}(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$ is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$.

7. Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that $\operatorname{curl} \mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$? Explain.

8. Show that any vector field of the form
$$\mathbf{F}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k}$$
is incompressible.

9. Show that any vector field of the form
$$\mathbf{F}(x, y, z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k},$$
where f, g, h are differentiable functions, is irrotational.

10. If $\mathbf{r}(x, y, z) = \langle x, y, z \rangle$ and $r = |\mathbf{r}|$, verify the identity:
    (a) $\nabla r = \mathbf{r}/r$
    (b) $\nabla \times \mathbf{r} = \mathbf{0}$
    (c) $\nabla(1/r) = -\mathbf{r}/r^3$
    (d) $\nabla \ln r = \mathbf{r}/r^2$

11. We have seen that all vector fields of the form $\mathbf{F} = \nabla g$ satisfy the equation $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and that all vector fields of the form $\mathbf{F} = \operatorname{curl} \mathbf{G}$ satisfy the equation $\operatorname{div} \mathbf{F} = 0$ (assuming continuity of the appropriate partial derivatives). This suggests the question: Are there any equations that all functions of the form $f = \operatorname{div} \mathbf{G}$ must satisfy? Show that the answer to this question is "No" by proving that *every* continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.