Worksheet 22, Math H53

Green's Theorem

Thursday, April 18, 2013

1. It appears as if Green's theorem tells us that

$$\int_C x\,dx = \iint_D 0\,dx\,dy = 0.$$

But we know from single-variable calculus that

$$\int x\,dx = \frac{x^2}{2} + C.$$

Is something amiss?

2. Evaluate the line integral $\oint_C (x-y)\,dx + (x+y)\,dy$ for C the circle with center the origin and radius 2, first directly, and then using Green's Theorem.

3. Use Green's Theorem to evaluate the line integral $\int_C xy^2\,dx + 2x^2y\,dy$ along the positively oriented boundary C of the triangle with vertices $(0,0)$, $(2,2)$, and $(2,4)$.

4. Use Green's Theorem to evaluate $\int_C y^4\,dx + 2xy^3\,dy$ for C the positively oriented boundary of the disk $x^2+y^2 \le 4$.

5. Use Green's Theorem to evaluate $\int_C \langle y\cos x - xy\sin x, xy + x\cos x\rangle \cdot d\mathbf{r}$ along the curve C, where C is the triangular path from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$. Make sure to check the orientation of the curve before applying the theorem.

6. A particle starts at the point $(-2,0)$, moves along the x-axis to $(2,0)$, and then along the semicircle $y=\sqrt{4-x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x,y) = \langle x, x^3+3xy\rangle$.

7. Let C be a closed curve. What geometric quantity is computed by

$$\frac{1}{2}\int_C -y\,dx + x\,dy?$$

There is a device used by surveyors called a mechanical integrator that uses this fact to find areas by tracing out boundaries.

8. Use the result of problem 7 to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1-\cos t$.

9. Calculate $\int_C \langle x^2+y, 3x-y^2\rangle \cdot d\mathbf{r}$, where C is the positively oriented boundary curve of a region D that has area 6.

10. If $\mathbf{F}(x,y) = \langle -y, x\rangle/(x^2+y^2)$, show that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.