Worksheet 19, Math H53
Multiple Integrals and Applications

Tuesday, April 9, 2013

1. Use polar coordinates to find the volume of the solid bounded by the parabooids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

2. Use cylindrical coordinates to find the volume that lies within both the cylinder $x^2 + y^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 16$.

3. Use spherical coordinates to evaluate $\iiint_H (9 - x^2 - y^2)\, dV$, where H is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$.

4. Find the mass and center of mass of the lamina that occupies the triangular region enclosed by the lines $x = 0$, $y = x$, and $2x + y = 6$, and has density function $\rho(x, y) = x^2$.

5. A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

6. Repeat the last problem, but now with the density at any point proportional to the square of its distance from the origin.

7. Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, is it more difficult to rotate the blade about the x-axis or the y-axis?

8. The joint probability density function for a pair of random variables X and Y is

$$f(x, y) = \begin{cases} Cx(1 + y) & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of the constant C.

   (b) Find $P(X \leq 1, Y \leq 1)$.

   (c) Find $P(X + Y \leq 1)$.

9. (a) Verify that

$$g(x, y) = \begin{cases} 4xy & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   is a joint probability density function.

   (b) If X and Y are random variables whose joint density function is g, find $P(X \geq 1/2)$ and $P(X \geq 1/2, Y \leq 1/2)$.

   (c) Find the expected values of X and Y.

10. (a) A lamp has two bulbs of a type with an average lifetime of 1000 hours. Assuming that we can model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$, find the probability that both of the lamp's bulbs fail within 1000 hours.

    (b) Another lamp has just one bulb of the same type as above, and if this bulb burns out, Jeeves the butler replaces it with another. Find the probability that at least two bulbs fail within a total of 1000 hours.