Worksheet 12, Math H53
Gradient Vectors and Directional Derivatives

Thursday, March 7, 2013

1. Find equations of the tangent plane and the normal line to the surface at the specified point:
   (a) $xyz^2 = 6$, $(3, 2, 1)$
   (b) $xy + yz + zx = 5$, $(1, 2, 1)$
   (c) $x^4 + y^4 + z^4 = 3x^2y^2z^2$, $(1, 1, 1)$
2. Find the directional derivative of the function $g(p, q) = p^4 - p^2q^3$ at the point $(2, 1)$ in the direction of the vector $\mathbf{v} = \langle 1, 3\rangle$.
3. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$.
   (a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.
   (b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points towards the origin.
4. If f and g are infinitely differentiable functions from $\mathbb{R}^3$ to $\mathbb{R}$, is the following a well-defined expression?
$$\nabla(1/(\nabla(f + g) \cdot ((\nabla f \times \nabla g) \times \nabla f)) + (\nabla(f + \sqrt[3]{g})f) \cdot \nabla g) \times (\nabla(fg) + \nabla(f/g))/2$$
5. Find the directional derivative of the function $f(x, y, z) = \sqrt{xyz}$ at the point $(3, 2, 6)$ in the direction of the vector $\mathbf{v} = \langle -1, -2, 2\rangle$.
6. For u and v differentiable functions of x and y, for a and b constants, and for f a differentiable function of one variable, show that the following properties hold for gradients:
   (a) $\nabla(au + bv) = a\nabla u + b\nabla v$
   (b) $\nabla(uv) = u\nabla v + v\nabla u$
   (c) $\nabla(u/v) = (v\nabla u - u\nabla v)/v^2$
   (d) $\nabla(f(u)) = f'(u)\nabla u$

   Thus gradients share some properties with standard 1-dimensional derivatives.
7. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.
8. Show that the equation of the tangent plane to the elliptic paraboloid $z/c = x^2/a^2 + y^2/b^2$ at the point (x_0, y_0, z_0) can be written as
$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = \frac{(z + z_0)/2}{c}$$
9. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f at this point? If so, how would you do it?
10. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.
11. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x}+\sqrt{y}+\sqrt{z} = \sqrt{c}$ is a constant.