Worksheet 10, Math H53

The Chain Rule

Thursday, February 28, 2013

1. Use the Chain Rule to find $\partial z/\partial s$
   (a) $z = \sin\theta\cos\phi$, $\theta = st^2$, $\phi = s^2 t$
   (b) $z = e^{x+2y}$, $x = s/t$, $y = t/s$
   (c) $z = \tan(u/v)$, $u = 2s + 3t$, $v = 3s - 2t$

2. Let $W(s,t) = F(u(s,t), v(s,t))$, where F, u, and v are differentiable, and

$$\begin{aligned} u(1,0) &= 2 & v(1,0) &= 3 \\ u_s(1,0) &= -2 & v_s(1,0) &= 5 \\ u_t(1,0) &= 6 & v_t(1,0) &= 4 \\ F_u(2,3) &= -1 & F_v(2,3) &= 10 \end{aligned}$$

   Find $W_s(1,0)$ and $W_t(1,0)$.

3. Assuming all functions are differentiable, use a tree diagram to write out the Chain Rule for the function $R = f(x,y,z,t)$, where $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$, and $t = t(u,v,w)$.

4. Use the chain rule to find the partial derivatives with respect to u, v, and w of $N = (p+q)/(p+r)$, where $p = u + vw$, $q = v + uw$, and $r = w + uv$.

5. Use the consequence of the Implicit Function Theorem (Equation 7 in Section 14.5 of Stewart 7e) to find $\partial z/\partial x$ and $\partial z/\partial y$ for the surface

$$x^2 + 2y^2 + 3z^2 = 1.$$

6. One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the area of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$?

7. If $z = f(x,y)$ is a differentiable function and $x = r\cos\theta$ and $y = r\sin\theta$, find $\partial z/\partial r$ and $\partial z/\partial\theta$, and show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

8. If $u = f(x,y)$ is a differentiable function and $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$

9. If $z = f(x-y)$ is a differentiable function, show that $\partial z/\partial x + \partial z/\partial y = 0$.

10. If $z = f(x,y)$ has continuous second-order partial derivatives, and $x = r\cos\theta$, $y = r\sin\theta$, find $\partial^2 z/\partial r\partial\theta$.

11. If f is homogeneous of degree n, that is, if it has continuous second order partial derivatives and satisfies the equation $f(tx,ty) = t^n f(x,y)$ for all t, show that

$$f_x(tx,ty) = t^{n-1} f_x(x,y).$$