Math 55 Quiz 2

September 7, 2016

This quiz will be graded out of 15 points; the True/False question is worth 3 points, and the exercise is worth 12 points. Please read the instructions carefully.

True or False. Mark the following statements as either true or false, or leave a blank if you don't know. A correct answer is worth +1 point, a blank is worth 0 points, and an incorrect answer is worth -1 points, so be smart about guessing!

a. F The proposition $\forall x, \exists y, P(x, y)$ is equivalent to the proposition $\exists y, \forall x, P(x, y)$.

b. T Proof by contradiction is a consequence of the inference rule *modus tollens*.

c. T Given propositions P_1, P_2, P_3, P_4, and P_5, we can show that all five of the propositions are equivalent by showing $P_1 \to P_4$, $P_2 \leftrightarrow P_5$, $P_3 \leftrightarrow P_4$, $P_4 \to P_5$, and $P_5 \to P_1$.

Exercise. A quantified logical proposition is said to be in *prenex normal form* if it is written so that it starts with a chain of quantifiers, and the remainder is a logical proposition which has no quantifiers. For instance, the following is in prenex normal form:

$$\underbrace{\forall \epsilon, \exists \delta, \forall x, \forall y,}_{\text{all quantifiers}} \underbrace{(\epsilon > 0 \land \delta > 0 \land |x - y| < \delta) \to |f(x) - f(y)| < \epsilon}_{\text{no quantifiers}}$$

However, the next statement is not:

$$(\forall x, P(x)) \lor (\forall y, Q(y))$$

Prove with a detailed chain of logical equivalences that the proposition

$$\forall x, (\exists y, P(x, y)) \to (\forall z, Q(x, z))$$

is equivalent to some proposition in prenex normal form.

Hint: The following equivalences may be useful:

a. $A \lor (\forall x, B(x)) \equiv \forall x, (A \lor B(x))$ b. $A \land (\forall x, B(x)) \equiv \forall x, (A \land B(x))$

c. $A \lor (\exists x, B(x)) \equiv \exists x, (A \lor B(x))$ d. $A \land (\exists x, B(x)) \equiv \exists x, (A \land B(x))$

(Since the problem statement is long, please use the back of this sheet for your solution.)

$$\forall x, (\exists y, P(x,y)) \to (\forall z, Q(x,z))$$

$$
\begin{aligned}
&\equiv \forall x, \neg(\exists y, P(x,y)) \vee (\forall z, Q(x,z)) && A \to B \equiv \neg A \vee B \\
&\equiv \forall x, (\forall y, \neg P(x,y)) \vee (\forall z, Q(x,z)) && \neg \exists y, A(y) \equiv \forall y, \neg A(y) \\
&\equiv \forall x, \forall z, ((\forall y, \neg P(x,y)) \vee Q(x,z)) && \text{a. above} \\
&\equiv \forall x, \forall z, (Q(x,z) \vee (\forall y, \neg P(x,y))) && A \vee B \equiv B \vee A \\
&\equiv \underline{\forall x, \forall z, \forall y,\ Q(x,z) \vee \neg P(x,y)} && \text{a. above.}
\end{aligned}
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