Worksheet 3, Math 54

Coordinates in Vector Spaces

Tuesday, March 4, 2014

1. True or false? Justify your answers.
   (a) The number of pivot columns of a marix equals the dimension of its column space.
   (b) A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.
   (c) The dimension of the vector space $\mathbb{P}_4$ is 4.
   (d) If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.
   (e) If a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly independent.
   (f) The row space of an $m \times n$ matrix A is the same as the column space of A^T.
   (g) If B is any echelon form of A, and if B has three nonzero rows, then the first three rows of A form a basis for the row space of A.
   (h) The dimensions of the row space and the column space of A are the same, even if A is not a square matrix.
   (i) The sum of the dimensions of the row space and the null space of A equals the number of rows in A.
   (j) On a computer, row operations can change the apparent rank of a matrix.
   (k) If $\dim V = n$ and if S is a set of vectors spanning V, then S is a basis of V.
   (l) The only 3-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.
   (m) A vector space is infinite-dimensional if and only if it is spanned by an infinite set of vectors, and it is not spanned by any finite set of vectors.

2. Use coordinate vectors to test the linear independence of the sets of polynomials:
   - $1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3$
   - $1 - 2t^2 - t^3, t + 2t^3, 1 + t - 2t^2$

3. Find a basis for the following subspace:

$$\left\{ \begin{bmatrix} p - 2q \\ 2p + 5r \\ -2q + 2r \\ -3p + 6r \end{bmatrix} : p, q, r \in \mathbb{R} \right\}$$

4. The first four "Hermite" polynomials are 1, $2t$, $-2 + 4t^2$, and $-12t + 8t^3$. These polynomials arise naturally in the study of certain important differential equations in mathematical physics. Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

5. Find the coordinates of the polynomial $p(t) = -1 + 8t^2 + 8t^3$ with respect to the basis of Hermite polynomials described in the previous question.

6. Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

7. How can the complex numbers $\mathbb{C}$ be considered as a real vector space? What is the dimension of $\mathbb{C}$, interpreted as such?

8. Let X be the set of infinite sequences of real numbers $a = (a_1, a_2, a_3, \ldots)$. How can you make sense of X as a real vector space? What is its dimension?

9. If A is a 5×4 matrix, what is the largest possible dimension of the row space of A? If A is a 4×5 matrix, what is the largest possible dimension of the row space of A? Explain.

10. If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A?

11. In statistics, a common requirement is that a matrix be of *full rank*, that is, the rank should be as large as possible. Explain why an $m \times n$ matrix with more rows than columns has full rank if and only if its columns are linearly independent.

12. Suppose A is an $m \times n$ matrix. Which of the subspaces $\operatorname{Row} A$, $\operatorname{Col} A$, $\operatorname{Nul} A$, $\operatorname{Row} A^T$, $\operatorname{Col} A^T$, and $\operatorname{Nul} A^T$ are in $\mathbb{R}^m$ and which are in $\mathbb{R}^n$? How many distinct subspaces are on this list?

13. Let $U = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $W = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for a vector space V, and let P be a matrix whose columns are $[\mathbf{u}_1]_W$ and $[\mathbf{u}_2]_W$. Which of the following equations is satisfied by P for all $\mathbf{x} \in V$?

    - $[\mathbf{x}]_U = P[\mathbf{x}]_W$
    - $[\mathbf{x}]_W = P[\mathbf{x}]_U$