Math 54 Quiz 7 Solutions

March 20th, 2014

1. Show that $\mathbf{a}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$ form an orthogonal set.

   Show that $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ form an orthogonal set.

   In $\mathbb{R}^3$, find the matrix of the transformation that reflects a vector $\mathbf{x}$ across the plane spanned by $\mathbf{b}_1, \mathbf{b}_2$, in the basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$

   Solution: To check that the vectors in these sets are orthogonal, we just need to take the dot product and see that it always gives zero. For the second part of the question, We note that in the basis $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, the matrix of the reflection transformation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. To get this matrix in the basis $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ we need to multiply on the right by the change of basis matrix from the $\mathbf{a}$ basis to the $\mathbf{b}$ basis and on the left by the change of basis matrix from the $\mathbf{b}$ basis to the $\mathbf{a}$ basis. If $\mathbf{A}$ is the matrix whose columns are $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ and $\mathbf{B}$ is the matrix with columns $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$, then the $\mathbf{a}$ to $\mathbf{b}$ change of basis matrix is $\mathbf{B^{-1}A}$, which is $\mathbf{B^TA}$ since the matrices are orthogonal, and the reverse change of basis matrix is the transpose of this, again since the matrices are orthogonal. So we need to calculate

   $\mathbf{A^TB} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{B^TA}$ which gives

   $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{bmatrix}$

2. Show that the product of two orthogonal matrices is orthogonal.
Solution: if $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, then $\mathbf{A^{-1}} = \mathbf{A^T}$ and similarly for $\mathbf{B}$, so $(\mathbf{AB})^\mathbf{T} = \mathbf{B^T A^T} = \mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$ so the product is also orthogonal.

3. True or False.

(a) Every square matrix has a complex eigenvalue.
True
(b) If $\mathbf{x}$ and $\mathbf{y}$ are vectors and $\mathbf{x} \cdot \mathbf{y} = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.
False, one of the vectors could be 0

(c) The product of two diagonalizable matrices is diagonalizable
False. Take for example $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}$

(d) Every 2×2 matrix with two distinct eigenvalues is diagonalizable
True. The eigenvectors for these two eigenvalues have to be linearly independent, so they form a basis of $\mathbb{R}^2$, hence the matrix must have a basis of eigenvectors and therefore be diagonalizable.