Math 54, Spring 2014
Midterm 2 Solutions
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1. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(a) Compute the rank of A.

Solution
A has only two distinct columns, which are not multiples of each other, so the dimension of the column space is 2. Thus the rank of A is 2.

(b) Find an orthonormal basis of the column space $\mathrm{Col}(A)$.

Solution
We apply Gram-Schmidt to the first two columns of A. If

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

then we take

$$\mathbf{v}_1 = \mathbf{u}_1, \quad \mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{\mathbf{v}_1} \mathbf{u}_2,$$

which gives

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

This set is orthogonal, but not orthonormal, so we scale the vectors to have length 1, which gives an orthonormal basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}.$$

(c) Find an orthonormal basis of the orthogonal complement of Col(A).

Solution
The orthogonal complement of Col(A) consists of the vectors orthogonal to this space, and is a 1-dimensional subspace of $\mathbb{R}^3$. Thus we need only find one vector orthogonal to the two distinct columns of A, and normalize it to length one. Thus we need

$$\mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

such that $x + y = 0$ and $y + z = 0$. We can easily see that such a vector is $x = z = 1$ and $y = -1$. Normalizing, we get a basis

$$\mathcal{C} = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}.$$

(d) Compute the 3×3 matrix P of the orthogonal projection onto Col(A).

Solution
Since $\mathcal{B}$ forms an orthonormal basis of Col(A), we have that the projection matrix can be obtained from the matrix B with columns consisting of the elements of $\mathcal{B}$ via

$$P = BB^{\mathrm{T}}.$$

Thus

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = 1/3 \cdot \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

(e) Show that your answers to b) and c) are eigenvectors of P and find an invertible matrix S so that $S^{-1}PS$ is diagonal.

Solution
An easy computation shows that $P\mathbf{x} = \mathbf{x}$ for $\mathbf{x} \in \mathcal{B}$, and that $P\mathbf{y} = 0$ for $\mathbf{y} \in \mathcal{C}$, so the elements of $\mathcal{B}$ are eigenvectors corresponding to eigenvalue $\lambda = 1$, and the element of $\mathcal{C}$ is an eigenvector corresponding to eigenvalue $\lambda = 0$.
The matrix S should be a matrix consisting of a basis of eigenvectors, which clearly are the vectors in $\mathcal{B}$ and $\mathcal{C}$:

$$S = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

(f) User your answer to d) to find a least squares solution to

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution
We solve the least squares problem by solving the system

$$A\mathbf{x} = P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix}.$$

Since any solution will do (the question only asks us to find *a* least squares solution), we note that the vector

$$\hat{\mathbf{x}} = \begin{bmatrix} 2/3 \\ -1/3 \\ 0 \\ 0 \end{bmatrix}$$

does the trick.

2. Check True or False.

(a) Gram-Schmidt applied to u, v, w or v, w, u yields the same set of orthogonal vectors.

Solution
False. Note that the first vector (possibly renormalized) is always in the output of the Gram-Schmidt algorithm, so we wouldn't expect the resulting set to be the same if we change which vector is first.

(b) If λ is an eigenvalue of a matrix B then λ^2 is an eigenvalue of B^2.

Solution
True. If $\mathbf{x}$ is an eigenvector of B with corresponding eigenvalue λ, then $\mathbf{x}$ is an eigenvector of B^2 with corresponding eigenvalue λ^2:

$$B^2\mathbf{x} = B(B\mathbf{x}) = B(\lambda\mathbf{x}) = \lambda B\mathbf{x} = \lambda^2\mathbf{x}.$$

(c) The matrices $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ are similar.

Solution
True. The matrices are similar via the relation

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

(d) The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ are similar.

Solution
False. Similar matrices have the same eigenvalues with the same degrees, but these matrices have different eigenvalues.

(e) If $U = U^{\mathrm{T}}$ and U is invertible then U is an orthogonal matrix.

Solution
False. For instance, the matrix

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

is a counterexample.

(f) If U is an orthogonal matrix then so is U^2.

Solution
True. A square matrix is orthogonal iff $UU^T = I_n$ the $n \times n$ identity matrix. Then if U is orthogonal,

$$(U^2)(U^2)^{\mathrm{T}} = UUU^{\mathrm{T}}U^{\mathrm{T}} = UI_nU^{\mathrm{T}} = UU^{\mathrm{T}} = I_n,$$

so U^2 is also orthogonal.

(g) The cosine of the angle of the vectors $(1, -1, -2)$ and $(2, -3, -1)$ is negative.

Solution
False. The dot product of vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = \|a\| \, \|b\| \cos(\theta),$$

where θ is the angle between the two vectors. In particular, since these norms are positive for nonzero vectors, the cosine of the angle between two vectors has the same sign as the dot product. Here, the dot product is $2 + 3 + 2 = 7 > 0$, so the cosine is positive.