Math 53 Quiz 11
April 26, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Find a parametric representation of the part of the cylinder $x^2 + z^2 = 4$ which lies between the planes $y + z = -1$ and $y + z = 1$.

We can use cylindrical coordinates with the cylinders oriented along the y-axis. Then fixing $r=2$ and allowing θ to vary freely, we can use the equations of the two bounding planes to give bounds on the values of y. Then the parametric representation is

$$\vec{r}(\theta, y) = \langle 2\cos\theta, y, 2\sin\theta \rangle,$$
$$0 \le \theta \le 2\pi, \quad -1 - 2\sin\theta \le y \le 1 - 2\sin\theta.$$

Problem 2. (3 points) Use Green's Theorem to evaluate the integral

$$\int_C (2x - y^3)\, dx + (x^3 + e^y)\, dy$$

where C is the positively oriented boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

The partial derivatives of the component functions $P(x,y) = 2x - y^3$ and $Q(x,y) = x^3 + e^y$ are continuous:

$$P_x = 2, \quad P_y = -3y^2, \quad Q_x = 3x^2, \quad Q_y = e^y,$$

so we can apply Green's Theorem to evaluate the integral as

$$\iint_D Q_x - P_y \, dA = \iint_D 3(x^2+y^2)\, dA = \int_0^{2\pi}\int_2^3 3r^2 \cdot r \, dr\, d\theta$$
$$= 6\pi \left(\frac{r^4}{4}\right)\Big|_{r=2}^{r=3} = \frac{195\pi}{2}.$$

Problem 3. (4 points) Evaluate the integral

$$\iint_S y^2\, dS$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

The cone $z = \sqrt{x^2+y^2}$ makes an angle of $\pi/4$ with the positive z-axis, so we can represent the surface using spherical coordinates by fixing $\rho = 1$ and allowing θ and ϕ to vary:

$$\vec{r}(\theta,\phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle,$$

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/4.$$

To reduce the integral to a standard multiple integral, we compute

$$\vec{r}_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle$$

$$\vec{r}_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle$$

$$\vec{r}_\theta \times \vec{r}_\phi = \langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi(\sin^2\theta + \cos^2\theta) \rangle$$

$$= -\sin\phi \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$

$$|\vec{r}_\theta \times \vec{r}_\phi| = |-\sin\phi| \cdot |\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle|$$

$$= |\sin\phi| = \sin\phi \quad (0 \le \phi \le \pi/4).$$

Thus the integral can be written as

$$\iint_S y^2\, dS = \int_0^{2\pi}\int_0^{\pi/4} (y(\theta,\phi))^2 \cdot |\vec{r}_\theta \times \vec{r}_\phi|\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \sin^3\phi \sin^2\theta\, d\phi\, d\theta$$

This integral is unreasonably tricky to evaluate, so full marks were awarded for this or an equivalent integral form, without evaluating.