Math 53 Quiz 10
April 19, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Evaluate the line integral

$$\int_C yz\cos(x)\,ds$$

where C is the helix described by $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$, $0 \le t \le \pi$.

We compute $|r'(t)|$ by

$$|r'(t)| = \sqrt{1^2+(-3\sin t)^2+(3\cos t)^2} = \sqrt{1^2+9(\cos^2 t+\sin^2 t)} = \sqrt{10}$$

Then we have

$$\int_C yz\cos(x)\,ds = \int_0^\pi (3\cos t)(3\sin t)\cos(t)\sqrt{10}\,dt$$

$$= 9\sqrt{10}\int_0^\pi \cos^2 t\sin t\,dt = -9\sqrt{10}\left(\frac{\cos^3 t}{3}\right)\Big|_0^\pi$$

$$= -9\sqrt{10}\left(\frac{(-1)^3-(1)^3}{3}\right) = \frac{-9\sqrt{10}\cdot(-2)}{3} = 6\sqrt{10}.$$

Problem 2. (3 points) Determine whether the vector field

$$\mathbf{F}(x,y) = \left\langle x\ln(1+x^2+y^2) + \frac{x^3}{1+x^2+y^2}, \frac{x^2y}{1+x^2+y^2}\right\rangle$$

is conservative.

If $\vec{F} = \langle P, Q\rangle$, then the partial derivatives given by

$$P_x = \ln(1+x^2+y^2) + \frac{2x^2}{1+x^2+y^2} + \frac{3x^2}{1+x^2+y^2} - \frac{2x^4}{(1+x^2+y^2)^2}$$

$$P_y = \frac{2xy}{1+x^2+y^2} - \frac{2x^3y}{(1+x^2+y^2)^2}$$

$$Q_x = \frac{2xy}{1+x^2+y^2} - \frac{2x^3y}{(1+x^2+y^2)^2}$$

$$Q_y = \frac{x^2}{1+x^2+y^2} - \frac{2x^2y^2}{(1+x^2+y^2)^2}$$

Since all of these derivatives are continuous on the open simply connected domain $\mathbb{R}^2$, and since $P_y = Q_x$ there, we can conclude that $\vec{F}$ is a conservative vector field.

Problem 3. (4 points) Use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = \langle \ln y + \cos x, x/y \rangle$$

and C is the curve parametrized by $\mathbf{r}(t) = \langle \pi t^2, 2t+1 \rangle$ for $0 \leq t \leq 1$.

If $\vec{F} = \nabla f$ for some function f, then we must have

$$\frac{\partial f}{\partial x} = \ln y + \cos x, \text{ and } \frac{\partial f}{\partial y} = x/y.$$

We can find f from this by working backwards. We have $f = \int \ln y + \cos x \, dx = x \ln y + \sin x + g(y)$ for some function g. Then from this we get that $\frac{\partial f}{\partial y} = x/y + g'(y)$, and since we already know that $\frac{\partial f}{\partial y} = x/y$, we see that $g'(y) = 0$, or $g(y) = C$ for a constant C.

Thus $f(x,y) = x \ln y + \sin x + C$.

To apply the fundamental theorem of line integrals, we compute the endpoints of the curve C as

$$\vec{a} = \vec{r}(0) = \langle 0, 1 \rangle, \text{ and } \vec{b} = \vec{r}(1) = \langle \pi, 3 \rangle.$$

Then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{b}) - f(\vec{a}) = \underline{\pi \ln 3}.$$