Math 53 Quiz 9

April 12, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Use the transformation $x = 5u$, $y = 3v$ to evaluate the integral $\iint_E y^2\,dA$ for the domain E bounded by the ellipse $9x^2 + 25y^2 = 900$.

By substituting in the equation of the ellipse, we see that the boundary of the region in the uv-plane corresponding to E is just

$$9(5u)^2 + 25(3v)^2 = 225u^2 + 225v^2 = 900$$
$$\longrightarrow u^2 + v^2 = 4.$$

Thus it is a circular domain, which we will call D. The Jacobian of the transformation is just $\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} = 15$, so we can compute:

$$\iint_E y^2\,dA = \iint_D 9v^2 \cdot 15\,dA = 135 \int_0^{2\pi}\int_0^2 (r\sin\theta)^2 \cdot r\,dr\,d\theta$$
$$= 135 \int_0^{2\pi} \sin^2\theta\,d\theta \int_0^2 r^3\,dr = 135\left(r^4/4\Big|_0^2\right)\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta = 540\pi.$$

Problem 2. (3 points) Express the Cartesian point $(x, y, z) = (\sqrt{3}, -1, 2\sqrt{3})$ in spherical coordinates.

The radius ρ is given by $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{3+1+12} = 4$.

The angle θ can be computed from x and y as in polar coordinates by $\theta = \tan^{-1}(y/x) = \tan^{-1}(-\sqrt{3}/3) = -\pi/6$.

Finally, we know that $z = \rho\cos\phi$, so $\cos\phi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, and we get that $\phi = \pi/6$.

Thus the spherical coordinates for the given point are

$$(\rho, \theta, \phi) = (4, -\pi/6, \pi/6).$$

Problem 3. (4 points) Use spherical or cylindrical coordinates to evaluate the integral

$$\iiint_E x^2 + y^2 + z^2 \, dV$$

for the region E bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$.

We'll integrate in cylindrical coordinates. The projection of the region E onto the xy-plane is the unit circle, so we can write the integral as

$$\iiint_E x^2+y^2+z^2 \, dV = \int_0^1 \int_0^{2\pi} \int_r^1 (r^2+z^2) r \, dz \, d\theta \, dr$$

$$= 2\pi \int_0^1 \int_r^1 (r^2+z^2) r \, dz \, dr = 2\pi \int_0^1 \left(r^3 z + r z^3/3\right)\Big|_{z=r}^{z=1} dr$$

$$= 2\pi \int_0^1 r^3 + r/3 - r^4 - r^4/3 \, dr = -\frac{2\pi}{3} \int_0^1 4r^4 - 3r^3 - r \, dr$$

$$= -2\pi/3 \left(4r^5/5 - 3r^4/4 - r^2/2\right)\Big|_0^1 = -2\pi/3 \left(4/5 - 3/4 - 1/2\right)$$

$$= 2\pi\left(5/12 - 4/15\right) = 3\pi/10.$$