Math 53 Quiz 9
April 12, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Express the point whose cylindrical coordinates are $(r, \theta, z) = (\sqrt{2}/2, \pi/2, -\sqrt{2}/2)$ in terms of spherical coordinates.

The value of θ is the same between the two coordinate systems, so we only need to find the values of ρ and ϕ. By comparing with rectangular coordinates, we see that $r^2 = x^2+y^2$ and $\rho^2 = x^2+y^2+z^2$, so we have that $\rho^2 = r^2+z^2 = 1$, and $\rho = 1$. Finally, we have $z = \rho\cos\phi$, so $\cos\phi = -\sqrt{2}/2$, and we find $\phi = 3\pi/4$. Thus the spherical coordinates for the point are

$$(\rho, \theta, \phi) = (1, \pi/2, 3\pi/4).$$

Problem 2. (3 points) Find the average distance from a point in a ball of radius a to its center.

Assume that the ball is centered at the origin. Then we want to find the average value of the function $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ on this ball. To do this, we compute

$$\bar{f} = \frac{1}{V(B)} \iiint_B \sqrt{x^2+y^2+z^2}\, dV = \frac{1}{V(B)} \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{2\pi}{V(B)} \int_0^{\pi} \sin\phi \, d\phi \int_0^{a} \rho^3 \, d\rho = \frac{2\pi}{V(B)} \left(-\cos\phi \Big|_0^{\pi}\right)\left(\frac{\rho^4}{4}\Big|_0^{a}\right)$$

$$= \frac{2\pi \cdot 2 \cdot a^4/4}{V(B)} = \frac{\pi a^4}{4/3\,\pi a^3} = \frac{3a}{4}.$$

Problem 3. (4 points) Let E be the region in the first quadrant of the plane bounded by the curves $y = 1/x$, $y = 4/x$, $y = x$, and $y = 3x$. Use the change of variables $x = \sqrt{u/v}$ and $y = \sqrt{uv}$ to compute

$$\iint_E xy\,dA$$

We can rewrite the boundary curves as

$$yx = 1, \quad yx = 4, \quad y/x = 1, \text{ and } y/x = 3.$$

Substituting with our change of variables in the two functions gives

$$yx = \sqrt{uv}\cdot\sqrt{u/v} = u, \quad y/x = \sqrt{uv}/\sqrt{u/v} = v.$$

Thus the domain in the uv-plane corresponding to the region E is just $1 \le u \le 4$ and $1 \le v \le 3$, which is rectangular. Computing the Jacobian, we have

$$\frac{\partial x,y}{\partial u,v} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{2}\cdot\sqrt{1/uv} & -\frac{1}{2}\cdot\sqrt{u/v^3} \\ \frac{1}{2}\cdot\sqrt{v/u} & \frac{1}{2}\cdot\sqrt{u/v} \end{vmatrix}$$

$$= \left(\tfrac{1}{2}\sqrt{1/uv}\right)\left(\tfrac{1}{2}\sqrt{u/v}\right) - \left(-\tfrac{1}{2}\sqrt{u/v^3}\right)\left(\tfrac{1}{2}\sqrt{v/u}\right)$$

$$= \tfrac{1}{4}\cdot\tfrac{1}{v} + \tfrac{1}{4}\cdot\tfrac{1}{v} = \tfrac{1}{2v}.$$

We can therefore rewrite the integral as

$$\iint_E xy\,dA = \int_1^4\int_1^3 u\cdot\tfrac{1}{2v}\,dv\,du = \tfrac{1}{2}\int_1^4 u\,du\int_1^3 \tfrac{1}{v}\,dv$$

$$= \tfrac{1}{2}\cdot\left(u^2/2\,\Big|_1^4\right)\left(\ln v\,\Big|_1^3\right) = \frac{15\ln 3}{4}.$$