Math 53 Quiz 8

April 5, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Find the average value of the function $f(x, y) = 1/\sqrt{x^2 + y^2}$ on the annular region $a^2 \leq x^2 + y^2 \leq b^2$, where $0 < a < b$.

The area of the region is the difference of the areas of two circles: $A = \pi b^2 - \pi a^2 = \pi(b^2 - a^2) = \pi(b-a)(b+a)$. Then the average value of f over this region can be computed with a polar integral by

$$\bar{f} = \frac{1}{A}\iint_D 1/\sqrt{x^2+y^2}\, dA = \frac{1}{A}\int_0^{2\pi}\int_a^b (1/r)\cdot r\, dr\, d\theta$$

$$= \frac{2\pi(b-a)}{A} = \frac{2}{b+a}.$$

Problem 2. (3 points) Rewrite the iterated integral

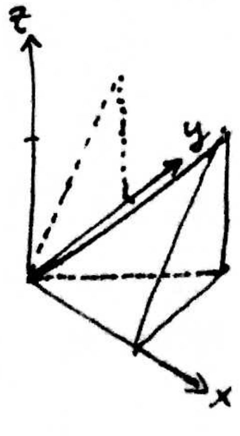
$$\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$$

as an iterated integral of the form $\iiint f(x, y, z)\, dx\, dy\, dz$.

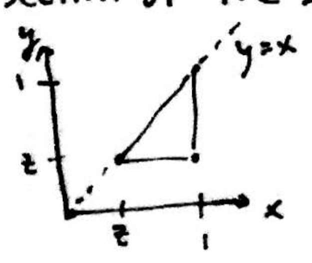
The region of integration is bounded by the surfaces

$y=0,\ y=1,\ x=y,\ x=1,\ z=0,\ z=y,$

so the region can be sketched as:



A z-cross section of the surface is given by:

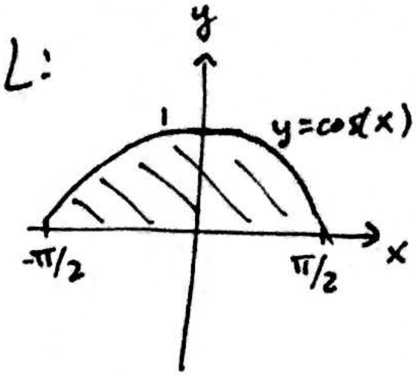


Thus we can rewrite the integral as $\displaystyle\int_0^1\int_z^1\int_y^1 f(x,y,z)\, dx\, dy\, dz$.

Problem 3. (4 points) Suppose that a lamina L occupies the region D bounded by the curves $y = 0$ and $y = \cos x$, $-\pi/2 \le x \le \pi/2$, and has density function $\rho(x, y) = y$. Find the mass of L, and write down (but don't evaluate!) expressions for the center of mass of L using iterated integrals.

The lamina is given by the following sketch.

L:



An integral over D can be written as an iterated integral of the form $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\cos(x)} f(x,y)\,dy\,dx$. In particular we have the mass of L:

$$m = \iint_D \rho(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos(x)} y\,dy\,dx = \int_{-\pi/2}^{\pi/2} \frac{\cos^2(x)}{2}\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2x)}{4}\,dx = \left(\frac{x}{4} + \frac{\sin(2x)}{8}\right)\Bigg|_{-\pi/2}^{\pi/2} = \pi/4.$$

Now if we let $M_y = \iint_D x\rho(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos(x)} xy\,dy\,dx$

and $M_x = \iint_D y\rho(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos(x)} y^2\,dy\,dx$,

then the center of mass has coordinates

$$\bar{x} = M_y/m, \quad \bar{y} = M_x/m.$$