Math 53 Quiz 6
March 15, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Find the directional derivative of the function $f(x, y, z) = xy + yz + zx$ at the point $P = (1, -1, 3)$ in the direction of the vector $\mathbf{v} = \langle 2, 3, 6 \rangle$.

To find the directional derivative, we compute the gradient vector ∇f at P, and the unit vector in the direction of $\vec{v}$.

$\nabla f = \langle y+z, x+z, y+x \rangle \longrightarrow \nabla f(1,-1,3) = \langle 2, 4, 0 \rangle$

$\vec{u} = \vec{v}/|\vec{v}| = \langle 2, 3, 6 \rangle / 7 = \langle 2/7, 3/7, 6/7 \rangle.$

Then the directional derivative is given by

$$D_{\vec{u}} f = \nabla f(1,-1,3) \cdot \vec{u} = \langle 2,4,0 \rangle \cdot \langle 2/7, 3/7, 6/7 \rangle = 16/7.$$

Problem 2. (3 points) Find and classify the local maximum and minimum points and the saddle points of $f(x, y) = y \sin x$.

To find the critical points of f, we find the solutions to

$\nabla f = \langle y \cos x, \sin x \rangle = \langle 0, 0 \rangle.$

$\sin x = 0$ when $x = k \cdot \pi$ for some integer k, and in particular for such x, $\cos x \neq 0$, so the first coordinate above gives us that $y = 0$. Thus the critical points of f are the points $(x,y) = (k\pi, 0)$ for k an arbitrary integer.

To classify these critical points we compute the Hessian determinant

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -y \sin x & \cos x \\ \cos x & 0 \end{vmatrix} = -\cos^2 x$$

For $(x,y) = (k\pi, 0)$, we get $D = -(\pm 1)^2 = -1 < 0$. This implies that all the critical points are saddle points.

Problem 3. (4 points) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find a parametric formula representing the *tangent line* to the ellipse at the point $(1, 2, 1)$.

The direction vector of the desired tangent line lies in both the tangent planes of the two surfaces at the point $(1,2,1)$, so it is given by the cross product of the normal vectors of the tangent planes. These can be computed using gradients.

$$f(x,y,z) = y+z=3 \rightarrow \nabla f = \langle 0,1,1\rangle \rightarrow \nabla f(1,2,1) = \langle 0,1,1\rangle.$$

$$g(x,y,z) = x^2+y^2=5 \rightarrow \nabla g = \langle 2x, 2y, 0\rangle \rightarrow \nabla g(1,2,1) = \langle 2,4,0\rangle.$$

The direction vector is then

$$\vec{v} = \langle 0,1,1\rangle \times \langle 2,4,0\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = \langle -4, 2, -2\rangle.$$

Thus the parametric formula of the tangent line is

$$\vec{r}(t) = \langle 1,2,1\rangle + t\langle -4,2,-2\rangle.$$