(a)

Math 53 Quiz 1

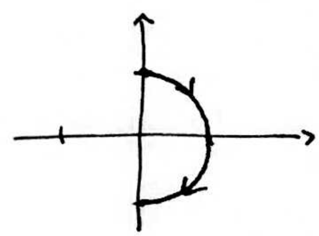
January 25, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Eliminate the parameter to find a Cartesian equation for the curve, and sketch the curve, indicating with an arrow the direction in which the curve is traced as the parameter increases.

$$x = 2\sin t\cos t, \quad y = \cos 2t, \quad 0 \leq t \leq \pi/2$$

A trig identity gives us that $x = 2\sin t\cos t = \sin 2t$, so in particular $x^2+y^2 = \sin^2 2t + \cos^2 2t = 1$, and the curve is a part of the unit circle. In particular we plug in a few values of t between 0 and $\pi/2$ to see that it starts at the point $(0,1)$, and travels clockwise to the point $(0,-1)$:



Problem 2. (3 points) Consider the curve given by

$$x = t^3 - 3t, y = t^3 - 1$$

Find the points on the curve where the tangent is horizontal or vertical. For which values of t is the curve concave upward?

First we compute $dx/dt = 3t^2 - 3 = 3(t+1)(t-1)$, and $dy/dt = 3t^2$. The curve has a vertical tangent when $dx/dt = 0$ and $dy/dt \neq 0$, so this is when $t = \pm 1$, at points $(2,-2)$ and $(-2,0)$. A horizontal tangent occurs when $dy/dt = 0$ and $dx/dt \neq 0$, in this case at $t=0$, or the point $(0,-1)$. The curve is concave up when $\frac{d^2y}{dx^2} \geq 0$, so we compute $dy/dx = (dy/dt)/(dx/dt) = \frac{t^2}{t^2-1}$, and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} = \frac{-2t/(t^2-1)^2}{3(t^2-1)} = \frac{-2}{3}\cdot\frac{t}{(t^2-1)^3}.$$

This is nonnegative for $t \leq -1$, and $0 \leq t \leq 1$.

Problem 3. (4 points) Set up (but don't evaluate) an integral that represents the length of the curve given by the parametrization

$$x = 5\cos(t/5), \quad y = 3\sin(t/3) + 2\cos(t/2)$$

The integral for the length of a curve is given by

$$\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

for bounds of integration chosen so that the parametrization traces the curve exactly once. The derivatives are easy to compute:

$$\frac{dy}{dt} = \cos(t/3) - \sin(t/2), \quad \frac{dx}{dt} = -\sin(t/5)$$

To find the right bounds of integration, we need to note that x is periodic mod 10π, while y is periodic mod 12π. Thus the curve repeats itself on the least common multiple of these periods, or every 60π. Thus the length is given by

$$L = \int_0^{60\pi} \sqrt{(\cos(t/3) - \sin(t/2))^2 + (-\sin(t/5))^2}\, dt.$$