Worksheet 13 Solutions, Math 53

Parametric Surfaces and Surface Integrals

Wednesday, November 28, 2012

1. Determine a parametric representation of the part of the sphere $x^2 + y^2 + z^2 = 16$ which lies above the cone $z = \sqrt{x^2 + y^2}$.

   Solution

   The cone specified forms an angle of $\pi/4$ with the z-axis, so we can represent the surface simply using spherical coordinates, by

   $$\mathbf{r}(\theta, \phi) = \langle 4\cos\theta\sin\phi, 4\sin\theta\sin\phi, 4\cos\phi\rangle, \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4.$$

2. Determine a parametric representation of a Möbius strip.

   Solution

   We think of the representation in two parts. First, we can think of the strip's center as following a circle in the xy-plane, which gives us a starting place. From there, we need an offset which traces out the width of the strip, with some amount of rotation depending on where on the central circle we are located. In order for the strip to form a Möbius strip, this radial component needs to rotate by just a half rotation when going around the circle.

   For the circle, we have a standard circle parametrization,

   $$\mathbf{r}_1(d, \theta) = \langle 2\cos(\theta), 2\sin(\theta), 0\rangle.$$

   Notice that this component completely ignores the parameter d.

   For the offset, we use the d parameter to give a location along the width of the strip. At $\theta = 0$, we assume that the strip lies flat, and so we should have an interval in the xy-plane. As θ increases, the strip should rotate in the z direction, so using a spherical coordinate representation with $\theta = \theta$, $\phi = \pi/2 - \theta/2$, and $\rho = d$, this gives us

   $$\mathbf{r}_2(d, \theta) = \langle d\cos(\theta)\cos(\theta/2), d\sin(\theta)\cos(\theta/2), d\sin(\theta/2)\rangle.$$

   This describes a line segment centered at the origin, which makes a polar angle of θ with the xz-plane, and a spherical angle of $\pi/2 - \theta/2$ with the z-axis. Because the angle with the z-axis only changes by π radians per 2π radians of change for θ, the segment only makes a half rotation in the direction of the z-axis per full 2π-period of θ.

   Together, we find that a full parametrization of the Möbius strip is given by:

   $$\begin{aligned}\mathbf{r}(d, \theta) &= \mathbf{r}_1(d, \theta) + \mathbf{r}_2(d, \theta) \\ &= \langle (2 + d\cos(\theta/2))\cos(\theta), (2 + d\cos(\theta/2))\sin(\theta), d\sin(\theta/2)\rangle, \quad -1 \le d \le 1,\ 0 \le \theta \le 2\pi.\end{aligned}$$

3. If the surface S is represented by $z = f(x, y)$ on the domain $x^2 + y^2 \leq R^2$, and you know that $|f_x| \leq 1$ and $|f_y| \leq 1$, then what can you say about the surface area of S?

Solution

The surface area is given by

$$S = \iint_S 1\, dS = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA,$$

so we may use the bound

$$1 = \sqrt{0+0+1} \leq \sqrt{(f_x)^2 + (f_y)^2 + 1} \leq \sqrt{1+1+1} = \sqrt{3}$$

to find that

$$1 \cdot A(D) \leq \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA \leq \sqrt{3} \cdot A(D).$$

Thus we see that S lies between πR^2 and $\sqrt{3}\pi R^2$.

4. Evaluate the surface integral $\iint_S y^2\, dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

Solution Idea

By drawing the cross-section of the surface in the xz-plane, we see that the line from the origin to the intersection of the cylinder with the sphere makes an angle of $\pi/6$ with the z-axis. Then it is simple to represent the surface using polar coordinates, with $\rho = 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi/6$. From here the integral is a standard exercise in integration techniques, using the formulation

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

5. Evaluate the surface integral $\iint_S \langle xy, 4x^2, yz \rangle \cdot d\mathbf{S}$, where S is the surface $z = xe^y$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$, with upward orientation.

Solution Idea

The parametrization is straightforward, since the surface is represented by a graph:

$$\mathbf{r}(x, y) = \langle x, y, xe^y \rangle, \quad 0 \leq x \leq 1,\ 0 \leq y \leq 1.$$

A normal vector is then given by $\mathbf{r}_x \times \mathbf{r}_y$, and to check the orientation, check to make sure that the z component of the resulting vector is positive. The computation reduces to standard integration by using

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA.$$

Alternatively, we have an equation specifically for the case of S a graph given by $z = g(x, y)$ with upward orientation, namely,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA.$$

6. Let $\mathbf{F}$ be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ for some constant c, where $\mathbf{r} = \langle x, y, z\rangle$. Show that the flux of $\mathbf{F}$ across a sphere S with center the origin is independent of the radius of S.

Solution

While one would be tempted, based on our recent homework problem 16.5.32 from Stewart's Calculus, 7th Edition, to try to use the Divergence Theorem in order to compute the flux in question, one would be immediately stymied by the fact that $\mathbf{F}$ is not defined at the origin. In fact, the divergence of $\mathbf{F}$ is zero everywhere except at the origin, but because of this single point, the Divergence Theorem fails to give us an alternative method of calculating the flux through a sphere around the origin.

However, one can still take this avenue of approach. Considering only the hollow ball $B(a, b)$ given in spherical coordinates by $a \leq \rho \leq b$ for positive $a < b$, we note that the surface of this region is given by the sphere $S_b : \rho = b$ with outward orientation, and the sphere $S_a : \rho = a$ with inward orientation. By arguing that the hollow ball can be represented as a finite union of simple solid regions, we can apply the divergence theorem to find that

$$\iint_{S_b} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B(a,b)} \operatorname{div} \mathbf{F}\, dV = \iiint_{B(a,b)} 0\, dV = 0,$$

and thus that

$$\iint_{S_b} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}.$$

From this, we see that the flux in question does not depend on the radius of the sphere. In fact, this argument can be extended to more complicated surfaces to show that the flux is actually the same for *any* reasonable surface around the origin.

In the case of spheres, a direct calculation will also demonstrate the desired result, and have the additional consequence of showing us the actual value of the flux in question.