Worksheet 12, Math 53
Green's Theorem, Divergence and Curl

Monday, November 19, 2012

1. It appears as if Green's theorem tells us that

$$\int_C x\,dx = \iint_D 0\,dx\,dy = 0.$$

   But we know from single-variable calculus that

$$\int x\,dx = \frac{x^2}{2} + C.$$

   Is something amiss?

2. Compute $\int_C y^2\,dx + x\,dy$ where C is the ellipse $x^2/a^2 + y^2/b^2 = 1$ oriented counter-clockwise.

3. Compute $\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$, where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$, with positive orientation.

4. Let C be a closed curve. What geometric quantity does the line integral

$$\frac{1}{2}\int_C -y\,dx + x\,dy$$

   compute?[1]

5. Let $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$.
   (a) Sketch $\mathbf{F}$ in the xy-plane.
   (b) Compute $\operatorname{curl}\mathbf{F}$ and include it in your previous sketch.
   (c) What is $\operatorname{curl}\mathbf{F}$ telling us about the fluid flow?

6. Let $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{G}(x, y) = -x\mathbf{i} - y\mathbf{j}$.
   (a) Sketch $\mathbf{F}$ and $\mathbf{G}$.
   (b) Compute $\operatorname{div}\mathbf{F}$ and $\operatorname{div}\mathbf{G}$.
   (c) For both $\mathbf{F}$ and $\mathbf{G}$, state if the origin is a fluid source or sink.

7. Determine whether or not the vector field $\mathbf{F}(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$ is conservative.

[1] There is a device used by surveyors called a *mechanical integrator* that uses this fact to find areas by tracing out boundaries.