Worksheet 12 Solutions, Math 53
Green's Theorem, Divergence and Curl

Monday, November 19, 2012

1. It appears as if Green's theorem tells us that

$$\int_C x\,dx = \iint_D 0\,dx\,dy = 0.$$

But we know from single-variable calculus that

$$\int x\,dx = \frac{x^2}{2} + C.$$

Is something amiss?

Solution

Nothing is amiss. The reason for this seemingly unintuitive result lies in the fact that the curve is closed. Working from the definitions, we see that the line integral $\int_C x\,dx$ actually gives us a definite integral of the form

$$\int_{x_0}^{x_1} x\,dx,$$

where x_0 is the x-coordinate of the starting position of the curve C, and x_1 is x-coordinate of the ending position. But since C needs to be closed in order for Green's theorem to apply, we see that x_1 is actually equal to x_0. This is what induces the supposed discrepancy:

$$\int_{x_0}^{x_1} x\,dx = \int_{x_0}^{x_0} x\,dx = 0.$$

2. Compute $\int_C y^2\,dx + x\,dy$ where C is the ellipse $x^2/a^2 + y^2/b^2 = 1$ oriented counter-clockwise.

Solution Idea

Green's Theorem translates the line integral into a double integral over the domain contained by the ellipse, and a change of variables $x = au$ and $y = bv$ translates the domain to a circle, giving a final value of πab.

3. Compute $\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$, where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$, with positive orientation.

Solution

Applying Green's Theorem gets rid of the funny-looking terms:

$$\begin{aligned}\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy &= \iint_D (2-1)\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 1\,dy\,dx \\ &= \int_0^1 \sqrt{x} - x^2\,dx = \left[2x^{3/2}/3 - x^3/3\right]\Big|_0^1 = 1/3\end{aligned}$$

4. Let C be a closed curve. What geometric quantity does the line integral

$$\frac{1}{2}\int_C -y\,dx + x\,dy$$

compute?[1]

Solution

Applying Green's theorem to the integral gives us that

$$\frac{1}{2}\int_C -y\,dx + x\,dy = \iint_D 1\,dA = A(D),$$

where D is the domain contained by the closed curve C. Thus this line integral actually computes the area contained by the curve C, using only information about the coordinates of the boundary points.

5. Let $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$.

   (a) Sketch $\mathbf{F}$ in the xy-plane.

   (b) Compute $\operatorname{curl}\mathbf{F}$ and include it in your previous sketch.

   (c) What is $\operatorname{curl}\mathbf{F}$ telling us about the fluid flow?

Solution Sketch

The sketch of $\mathbf{F}$ in the xy-plane consists of arrows perpendicular to a line to the origin, pointing in a counterclockwise direction, with length equal to the distance from the origin.

A quick computation shows that the curl is constant equal to $\langle 0, 0, 2\rangle$, which indicates that an infinitesimal parcel of fluid located at a point in this fluid flow is rotating at a rate of 2 radians per second in the counterclockwise direction in the xy-plane.

6. Let $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{G}(x, y) = -x\mathbf{i} - y\mathbf{j}$.

   (a) Sketch $\mathbf{F}$ and $\mathbf{G}$.

   (b) Compute $\operatorname{div}\mathbf{F}$ and $\operatorname{div}\mathbf{G}$.

   (c) For both $\mathbf{F}$ and $\mathbf{G}$, state if the origin is a fluid source or sink.

Solution Sketch

The sketch of $\mathbf{F}$ consists of arrows pointed away from the origin of length equal to the distance from the base point to the origin, while the sketch of $\mathbf{G}$ consists of arrows pointed directly to the origin, in particular with length equal to the length from the base point to the origin.

Some quick computations show that $\operatorname{div}\mathbf{F}(x, y) = 2$, and $\operatorname{div}\mathbf{G}(x, y) = -2$, i.e. that the divergence for both vector fields is constant for all points (x, y). The sketches then indicate that the origin is a fluid source for $\mathbf{F}$ and a fluid sink for $\mathbf{G}$.

[1] There is a device used by surveyors called a *mechanical integrator* that uses this fact to find areas by tracing out boundaries.

7. Determine whether or not the vector field $\mathbf{F}(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$ is conservative.

   Solution

   The partial derivatives of $\mathbf{F}$ are enumerated by the Jacobian matrix

$$J_{\mathbf{F}} = \begin{bmatrix} \partial P/\partial x & \partial P/\partial y & \partial P/\partial z \\ \partial Q/\partial x & \partial Q/\partial y & \partial Q/\partial z \\ \partial R/\partial x & \partial R/\partial y & \partial R/\partial z \end{bmatrix} = \begin{bmatrix} 0 & ze^{yz} & ye^{yz} \\ ze^{yz} & xz^2e^{yz} & xe^{yz} + xyze^{yz} \\ ye^{yz} & xe^{yz} + xyze^{yz} & xy^2e^{yz} \end{bmatrix},$$

   and from this we can see that all of the partial derivatives are continuous. Computing the curl of $\mathbf{F}$ gives us

$$\operatorname{curl} \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \langle 0, 0, 0 \rangle .$$

   As we have seen, if all partial derivatives are continuous and the curl is zero, then the vector field is conservative.