Worksheet 11 Solutions, Math 53
Line Integrals

Wednesday, November 7, 2012

1. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \le t \le b$, and $\mathbf{v}$ is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

Solution

Suppose that $\mathbf{v} = \langle x_0, y_0, z_0 \rangle$. Then $f(x, y, z) = x_0 x + y_0 y + z_0 z$ is clearly a function such that $\mathbf{v} = \nabla f$. In particular, we see that $\mathbf{F}(x, y, z) = \mathbf{v}$ the constant vector field with value $\mathbf{v}$ is conservative, and so the Fundamental Theorem of Line Integrals applies. But $f(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ for any vector $\mathbf{x}$, so we see that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = \mathbf{v} \cdot \mathbf{r}(b) - \mathbf{v} \cdot \mathbf{r}(a) = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)],$$

as desired.

2. Determine whether or not $\mathbf{F}(x, y) = e^x \cos y\,\mathbf{i} + e^x \sin y\,\mathbf{j}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

Solution

The Jacobian matrix of $\mathbf{F}$ enumerates the partial derivatives of the components of $\mathbf{F}$, and is given by

$$J_{\mathbf{F}} = \begin{bmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

If $\mathbf{F}$ were the gradient of some function f, then it would be conservative, and since all of its partial derivatives are continuous, we would have $\partial P/\partial y = \partial Q/\partial x$. Since this is clearly not the case, we can conclude that there is no function f such that $\mathbf{F} = \nabla f$.

3. Suppose you're asked to determine the curve that requires the least work for a force field $\mathbf{F}$ to move a particle from one point to another. You decide to check first whether $\mathbf{F}$ is conservative, and indeed it turns out that it is. How would you reply to the request?

Solution

Since $\mathbf{F}$ is conservative, the line integral $\int_C \mathbf{F}\, d\mathbf{r}$ depends only on the endpoints of C, and not on the geometry in between. Since this integral can be interpreted as the work done by $\mathbf{F}$ on a particle moving through the force field, it doesn't matter which curve is used.

4. Evaluate the line integral

$$\int_C z^2\,dx + x^2\,dy + y^2\,dz,$$

where C is the line segment from $(1,0,0)$ to $(4,1,2)$.

Solution

We parametrize C by

$$\mathbf{r}(t) = (1-t)\langle 1,0,0\rangle + t\langle 4,1,2\rangle = \langle 3t+1, t, 2t\rangle\,, \quad 0 \le t \le 1.$$

Then $\mathbf{r}'(t) = \langle 3,1,2\rangle$, and the integral can be rewritten as

$$\begin{aligned}
\int_C z^2\,dx + x^2\,dy + y^2\,dz &= \int_C \left\langle z^2, x^2, y^2\right\rangle \cdot d\mathbf{r} \\
&= \int_0^1 \left\langle (2t)^2, (3t+1)^2, (t)^2\right\rangle \cdot \langle 3,1,2\rangle\ dt \\
&= \int_0^1 3(4t^2) + (9t^2+6t+1) + 2(t^2)\,dt \\
&= \int_0^1 23t^2 + 6t + 1\,dt \\
&= \left(23t^3/3 + 3t^2 + t\right)\Big|_0^1 = 35/3.
\end{aligned}$$

5. Evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r},$$

where $\mathbf{F}(x,y) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$, and C is given by the vector function $\mathbf{r}(t) = (t+\sin(\frac{1}{2}\pi t))\,\mathbf{i} + (t + \cos(\frac{1}{2}\pi t))\,\mathbf{j}$, with $0 \le t \le 1$.

Solution

This is a terrible integral to solve directly, so we apply the fundamental theorem of line integrals. The Jacobian matrix of $\mathbf{F}$ enumerates the partial derivatives of the components of $\mathbf{F}$, and is given by

$$J_{\mathbf{F}} = \begin{bmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{bmatrix} = \begin{bmatrix} y^2 & 2xy \\ 2xy & x^2 \end{bmatrix}.$$

From this we can see that all of the partial derivatives of $\mathbf{F}$ are continuous, and that $\partial P/\partial y = \partial Q/\partial x$, so we can conclude that $\mathbf{F}$ is a conservative vector field.

A straightforward calculation shows that $f(x,y) = x^2y^2/2$ is a function whose gradient is $\langle P, Q\rangle$, so the Fundamental Theorem of Line Integrals gives us that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2,1) - f(0,1) = 2.$$

6. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F}(x, y, z) = \sin x\, \mathbf{i} + \cos y\, \mathbf{j} + xz\, \mathbf{k}$, and C is given by the vector function $\mathbf{r}(t) = t^3\, \mathbf{i} - t^2\, \mathbf{j} + t\, \mathbf{k}$, with $0 \leq t \leq 1$.

Solution

This integral can be approached directly using definitions. The derivative of $\mathbf{r}(t)$ is given by $\mathbf{r}'(t) = \left\langle 3t^2, -2t, 1 \right\rangle$, and so we can calculate

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^1 \left\langle \sin(t^3), \cos(-t^2), t^4 \right\rangle \cdot \left\langle 3t^2, -2t, 1 \right\rangle\, dt \\
&= \int_0^1 3t^2 \sin(t^3) - 2t\cos(-t^2) + t^4\, dt \\
&= \int_0^1 3t^2 \sin(t^3)\, dt - \int_0^1 2t\cos(t^2)\, dt + \int_0^1 t^4\, dt \\
&= \int_0^1 \sin(u)\, du - \int_0^1 \cos(v)\, dv + \int_0^1 t^4\, dt \\
&= -\cos(u)\big|_0^1 - \sin(v)\big|_0^1 + t^5/5\big|_0^1 \\
&= 6/5 - \cos(1) - \sin(1).
\end{aligned}$$