Worksheet 7 Solutions, Math 53
Gradients and Optimization

Wednesday, October 10, 2012

1. Find the equation of the tangent plane and the normal line to the surface

$$x + y + z = e^{xyz}$$

at the point $(0, 0, 1)$.

Solution

We can represent this surface by $f(x, y, z) = x + y + z - e^{xyz} = 0$, so computing a gradient vector gives us

$$\nabla f = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle.$$

At $(0, 0, 1)$ this is just $\langle 1, 1, 1 \rangle$, and so we have that the tangent plane is given by

$$x + y + (z - 1) = 0,$$

and the normal line is given by

$$\mathbf{x}(t) = \langle 0, 0, 1 \rangle + \langle 1, 1, 1 \rangle t.$$

2. Find the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point (x_0, y_0, z_0).

Solution

The gradient of $f(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$ at (x_0, y_0, z_0) is given by

$$\nabla f = 2 \left\langle \frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2} \right\rangle,$$

so the tangent plane is given by

$$\frac{(x - x_0)x_0}{a^2} + \frac{(y - y_0)y_0}{b^2} + \frac{(z - z_0)z_0}{c^2} = 0,$$

or

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1.$$

3. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r, using both the method of derivative tests and the method of Lagrange multipliers.

Solution Idea

For the derivative tests method, assume that the sphere is centered at the origin, and consider the circular projection of the sphere onto the xy-plane. An inscribed rectangular box is uniquely determined

by the xy-coordinate of its corner in the first octant, so we can compute the z coordinate of this corner by
$$x^2 + y^2 + z^2 = r^2 \implies z = \sqrt{r^2 - (x^2 + y^2)}.$$
Then the volume of a box with this coordinate for the corner is given by
$$V = (2x)(2y)(2z) = 8xy\sqrt{r^2 - (x^2 + y^2)},$$
and we need only maximize this on the domain $x^2 + y^2 \le r^2$. Notice that the volume is zero on the boundary of this domain, so we need only consider critical points contained inside the domain in order to carry this optimization out.

For the method of Lagrange multipliers, we optimize $V(x, y, z) = 8xyz$ subject to the constraint $x^2 + y^2 + z^2 = r^2$.

4. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin, using both the method of derivative tests and the method of Lagrange multipliers.

   Solution Idea

   Since y appears in this equation only as a square, any point (x, y, z) on the surface has a corresponding solution $(x, -y, z)$, and we can assume without loss of generality that y is non-negative (recalling in the end that there will be a corresponding point with y non-positive which is an equal distance from the origin). In this case, we can let $y = \sqrt{9 + xz}$, and we can optimize the function $D(x, z) = \sqrt{x^2 + (9 + xz) + z^2}$ in the xz-plane.

   For the method of Lagrange multipliers, we optimize $D(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ subject to the constraint $y^2 - xz = 9$. Note that we may also optimize $D^2(x, y, z) = x^2 + y^2 + z^2$ the square distance from the origin with the same result.

5. Find the extreme values of $f(x, y) = x^2 + y^2 + 4x - 4y$ on the region $x^2 + y^2 \le 9$.

   Solution

   ∇f is given by $\langle 2x + 4, 2y - 4 \rangle$, so the only critical point for this function is at $(x, y) = (-2, 2)$, which is contained in the region in question.

   To test for potential extreme values on the boundary of the region, we use Lagrange multipliers. Setting $g(x, y) = x^2 + y^2$, we get the system of equations
   $$\begin{cases} 2x + 4 = \lambda 2x \\ 2y - 4 = \lambda 2y \\ x^2 + y^2 = 9 \end{cases}$$

   Solving this system gives us two possible values for (x, y), namely $(3/\sqrt{2}, -3/\sqrt{2})$ and $(-3/\sqrt{2}, 3/\sqrt{2})$. Plugging all of the possible extreme values into f gives us
   $$f(-2, 2) = -8, \quad f(3/\sqrt{2}, -3/\sqrt{2}) = 9 + 12\sqrt{2} \approx 25.97, \quad f(-3/\sqrt{2}, 3/\sqrt{2}) = 9 - 12\sqrt{2} \approx -7.97$$
   Thus the minimum of -8 occurs at $(-2, 2)$, and the maximum of $9 + 12\sqrt{2}$ occurs at $(3/\sqrt{2}, -3/\sqrt{2})$.

6. Find the maximum value of
   $$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$
   given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant. Deduce that for positive $x_1, x_2, \ldots, x_n$ we have
   $$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

This means the geometric mean of n numbers is no larger than the arithmetic mean of these numbers. When are these two means equal?

Solution Sketch

Lagrange multipliers on f subject to $g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ gives us that the largest value of f occurs at $(x_1, x_2, \ldots, x_n) = (c/n, c/n, \ldots, c/n)$, where it takes on value c/n. But then we may just read off the identity we seek by noticing for any value of c,

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{c}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

In particular, because the legrange multipliers gave us exactly one point at which this maximum value is achieved, we can conclude that for every other choice of $(x_1, x_2, \ldots, x_n)$, we must have a strict inequality, so the two means are equal only when each of the numbers being averaged is equal.