

Worksheet 6 Solutions, Math 53

Tangent Planes and Gradient Vectors

Wednesday, October 3, 2012

1. Let S be the surface represented by the equation $xy + yz + zx = 5$, and let $P(1, 2, 1)$ be a point on S . Find an equation of the tangent plane of S at P .

Solution

Given a function f , ∇f gives us a vector perpendicular to the level surface of f at any given point. Thus letting $f(x, y, z) = xy + yz + zx$, we compute

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle = \langle y + z, x + z, x + y \rangle.$$

Then at $(1, 2, 1)$ this gives us a perpendicular vector of $\nabla f(1, 2, 1) = \langle 3, 2, 3 \rangle$, and so the tangent plane of S at P is given by

$$3(x - 1) + 2(y - 2) + 3(z - 1) = 0.$$

2. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. You don't have an equation for S , but you know that the curves

$$\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle$$

$$\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle$$

both lie on S . Find an equation of the tangent plane of S at P .

Solution

As long as \mathbf{r}_1 and \mathbf{r}_2 are not tangent at P , their derivatives give us two non-parallel vectors which lie in the tangent plane, and thus which determine a unique direction for the perpendicular vector.

To compute a perpendicular vector, we calculate the derivatives of \mathbf{r}_1 and \mathbf{r}_2 at P .

$$\mathbf{r}'_1(t) = \langle 3, -2t, 2t - 4 \rangle$$

$$\mathbf{r}_1(t) = \langle 2, 1, 3 \rangle \implies t = 0$$

$$\mathbf{r}'_1(0) = \langle 3, 0, -4 \rangle$$

$$\mathbf{r}'_2(u) = \langle 2u, 6u^2, 2 \rangle$$

$$\mathbf{r}_2(u) = \langle 2, 1, 3 \rangle \implies u = 1$$

$$\mathbf{r}_2(1) = \langle 2, 6, 2 \rangle$$

Then a vector perpendicular to these two vectors is given by their cross product:

$$\langle 3, 0, -4 \rangle \times \langle 2, 6, 2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = \langle 0 + 24, -8 - 6, 18 - 0 \rangle = \langle 24, -14, 18 \rangle.$$

Thus the perpendicular plane to the surface S at point P can be expressed as

$$24(x - 2) - 14(y - 1) + 18(z - 3) = 0.$$

3. Show that the sum of the x -, y -, and z -intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

Solution

To find the tangent plane of this surface, we compute the gradient vector of $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$:

$$\nabla f = \left\langle \frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}}, \frac{1}{2\sqrt{z}} \right\rangle,$$

and we write the equation for the tangent plane at (x_0, y_0, z_0) as

$$\frac{x - x_0}{2\sqrt{x_0}} + \frac{y - y_0}{2\sqrt{y_0}} + \frac{z - z_0}{2\sqrt{z_0}} = 0.$$

But in fact, this can be rewritten as

$$\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c},$$

where the last equality follows because we are assuming that (x_0, y_0, z_0) lies on the surface. Then the intercepts with the coordinate axes can be determined by substituting zero for two of the variables x , y , and z , and this gives us intercepts of $\sqrt{x_0}\sqrt{c}$, $\sqrt{y_0}\sqrt{c}$, and $\sqrt{z_0}\sqrt{c}$, which means that the sum of these intercepts is just

$$(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0})\sqrt{c} = \sqrt{c}\sqrt{c} = c.$$

4. Find the maximum rate of change of $f(x, y) = \sin(xy)$ at the point $(x, y) = (1, 0)$, and determine the direction in which this rate of change is achieved.

Solution

Here we use the nice properties of the gradient vector, namely that the direction of this vector is the direction in which a function is increasing most rapidly, and the length of the vector is the rate of change in this direction. We have

$$\nabla f = \langle y \cos(xy), x \cos(xy) \rangle,$$

so at the point $(1, 0)$ this gives us $\nabla f(1, 0) = \langle 0, 1 \rangle$, which means that the function is increasing most rapidly in the positive y direction, at a rate of 1.

5. Two surfaces are called *orthogonal* at a point of intersection if their normal lines are perpendicular at that point.

- (a) Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{at } P$$

Solution Idea

This follows because the gradient vector is perpendicular to a level surface, and two vectors are orthogonal if and only if their dot product is zero.

- (b) Use this fact to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection.

Solution

Using the notation from the previous problem, we can let $F(x, y, z) = x^2 + y^2 - z^2$, and $G(x, y, z) = x^2 + y^2 + z^2 - r^2$. Then we have

$$\nabla F = \langle 2x, 2y, -2z \rangle, \quad \nabla G = \langle 2x, 2y, 2z \rangle,$$

and so

$$F_x G_x + F_y G_y + F_z G_z = 4(x^2 + y^2 - z^2).$$

But one of the surfaces is given by $x^2 + y^2 - z^2 = 0$, and so any intersection point must lie on this surface, and hence we have that the above dot product is zero. From this we can conclude that the surfaces are perpendicular.