Worksheet 5 Solutions, Math 53

Limits and Derivatives in Multiple Dimensions

Wednesday, September 26, 2012

1. Determine the set of points at which the function is continuous:

   (a) $f(x, y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$

   Solution

   Since f is a rational function, it is continuous on its entire domain, or where $1 - x^2 - y^2 \neq 0$. This is all of $\mathbb{R}^2$ except for the unit circle.

   (b) $f(x, y) = \tan^{-1}\left(\left(\dfrac{1}{x+y}\right)^2\right)$

   Solution

   $1/(x + y)^2$ is rational, and hence is continuous in its domain, which is everywhere except the line $y = -x$. $\tan^{-1}$ is continuous on $\mathbb{R}$, so we see that the entire function is continuous except on the line $y = -x$.

   (c) $f(x, y) = \begin{cases} \dfrac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

   Solution

   On $\mathbb{R}^2$ without the origin, f is a rational function, and so is continuous at every point in its domain. Since it is defined on this entire domain, we see that f is continuous at every point except the origin.

   To check whether f is continuous at the origin, we must check whether

   $$\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0).$$

   Checking along the line $y = 0$, we see that

   $$\lim_{x\to 0} \frac{x^2(0)^3}{2x^2 + (0)^2} = \lim_{x\to 0} 0 = 0.$$

   But this implies that if f has a limit as (x, y) tends to $(0, 0)$, it must be 0, which is not the value of $f(0, 0)$. Thus we see that in this case, $f(x, y)$ is not continuous at the origin.

2. If $\mathbf{c}$ is a fixed vector in $\mathbb{R}^n$, show that the function f given by $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is continuous on $\mathbb{R}^n$.

   Solution

   If $\mathbf{c} = \mathbf{0}$, then f is the constant zero function, and so is trivially continuous. So assume that $\mathbf{c} = \langle c_1, c_2, \ldots, c_n\rangle \neq \mathbf{0}$, and let $C = \max_i |c_i|$. We want to show that for any vector $\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$, we have

   $$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

To see that this is the case, notice first that $f(\mathbf{x}) + f(\mathbf{y}) = f(\mathbf{x}+\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$. Then writing $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we have

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{a})| = |f(\mathbf{x}-\mathbf{a})| &= \left| \sum_{i=1}^{n} c_i (x_i - a_i) \right| \\ &\leq \sum_{i=1}^{n} |c_i| \, |x_i - a_i| \leq \sum_{i=1}^{n} C \, |x_i - a_i| \leq \sum_{i=1}^{n} C \, \|\mathbf{x}-\mathbf{a}\| = nC \, \|\mathbf{x}-\mathbf{a}\| \end{aligned}$$

Here we have applied the triangle inequality, the definition of C, and the inequality

$$|x_i - a_i| \leq \|\mathbf{x}-\mathbf{a}\|, \quad 1 \leq i \leq n,$$

which holds true when you compare the components of any vector with the magnitude of that vector. In particular, using this inequality, we see that if $\epsilon > 0$ is given, then requiring that

$$\|\mathbf{x}-\mathbf{a}\| < \epsilon/(nC)$$

is sufficient in order to have $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$. That is, in the definition of continuity, we can choose $\delta = \epsilon/(nC)$. Since $\mathbf{a}$ was chosen arbitrarily, this shows that f is continuous at any point in $\mathbb{R}^n$.

Alternatively, we may note that $f(\mathbf{x})$ is a polynomial in n variables, and so is continuous everywhere.

3. If $f(x,y,z) = \sqrt{1+xz} + \sqrt{1-xy}$, find f_{xyz}. Can you think of a way to do this computation in your head?

   Solution

   We can write $f(x,y,z) = g(x,z) + h(x,y)$, where $g(x,z) = \sqrt{1+xz}$ and $h(x,y) = \sqrt{1-xy}$. Thus in particular, we notice that by linearity of partial derivatives,

   $$f_{xyz} = g_{xyz} + h_{xyz}.$$

   But g is constant with respect to y, and h is constant with respect to z, so both of these derivatives are actually 0, meaning $f_{xyz}(x,y,z) = 0$.

4. A friend tells you that there is a function f whose partial derivatives are $f_x(x,y) = x + 4y$ and $f_y(x,y) = 3x - y$. Is your friend a dirty liar?

   Solution

   Taking further partial derivatives of these supposed partial derivatives gives us that the second partial derivatives are defined and continuous everywhere, and so in particular should be equal by Clairaut's Theorem. However, we find that according to your friend,

   $$f_{xy}(x,y) = 4, \qquad f_{yx}(x,y) = 3,$$

   which contradicts our previous conclusion. We see that your friend must be pulling your leg.

5. If a, b, c are the sides of a triangle, and A, B, C are the angles opposite the respective sides, find $\partial A/\partial a$, $\partial A/\partial b$, and $\partial A/\partial c$ by implicit differentiation of the law of cosines.

   Solution Sketch

   The law of cosines gives us that $a^2 = b^2 + c^2 - 2bc\cos(A)$. Considering A as a function of a, b and c, we use implicit differentiation to find that

$$\frac{\partial A}{\partial a} = \frac{a\csc(A)}{bc}$$
$$\frac{\partial A}{\partial b} = \frac{c\cot(A) - b\csc(A)}{bc}$$
$$\frac{\partial A}{\partial c} = \frac{b\cot(A) - c\csc(A)}{bc}$$

6. How many nth-order partial derivatives does a function of two variables have? If these partial derivatives are all continuous, how many of them can be distinct?

   Solution

   For each of the successive partial derivatives, we can choose either to take an x derivative or a y derivative, so we have n choices of two possibilities, which gives a total count of 2^n partial derivatives.

   Assuming (correctly) that Clairaut's theorem can be generalized to arbitrary numbers of derivatives, if all of the partial derivatives are continuous, then the value of the derivatives depends only on the number of x derivatives taken and the number of y derivatives taken. Since there are only $n + 1$ possible choices for these numbers, this means that there can only be $n + 1$ distinct functions in the entire collection of 2^n partial derivatives.