Worksheet 4 Solutions, Math 53

Vector Geometry and Vector Functions

Monday, September 17, 2012

1. Find an equation of the plane:

   (a) The plane through the point $(2, 4, 6)$ and parallel to the plane $z = x + y$.

   Solution
   Rewriting the original plane in the standard form gives the equation $x + y - z = 0$, so we have a normal vector of $\langle 1, 1, -1 \rangle$ and a base point of $(2, 4, 6)$, which gives us the equation

   $$(x - 2) + (y - 4) - (z - 6) = 0.$$

   (b) The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$.

   Solution
   We can compute a normal vector by taking the cross product of two vectors in the plane, say the vector $\mathbf{a}$ from $(0, 1, 1)$ to $(1, 0, 1)$ and the vector $\mathbf{b}$ from $(0, 1, 1)$ to $(1, 1, 0)$. This gives us $\mathbf{a} = \langle 1, -1, 0 \rangle$ and $\mathbf{b} = \langle 1, 0, -1 \rangle$, and so the cross product is

   $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle .$$

   Using the base point $(0, 1, 1)$, this gives us the equation

   $$x + (y - 1) + (z - 1) = 0.$$

   (c) The plane that passes through the point $(1, -1, 1)$ and contains the line with symmetric equations $x = 2y = 3z$.

   Solution Idea
   The line in question clearly contains the points $(0, 0, 0)$ and $(6, 3, 2)$. (Picking a value for one of the coordinates determines the rest.) From here, we have three points in the plane, and the equation may be determined using the same method as in the previous part.

2. Where does the line through $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$?

   Solution
   We first represent the line as a vector-valued function, and then solve for the intersection point. A vector $\mathbf{a}$ in the direction of the line is given by the vector between $(1, 0, 1)$ and $(4, -2, 2)$, so we can let $\mathbf{a} = \langle 3, -2, 1 \rangle$. Then denoting $\mathbf{b} = \langle 1, 0, 1 \rangle$, a vector function for the line is

   $$\mathbf{r}(t) = \mathbf{b} + \mathbf{a}t = \langle 3t + 1, -2t, t + 1 \rangle , \quad t \in \mathbb{R}.$$

   Then the point at which the line intersects the plane is just the point at which the coordinates satisfy the equation of the plane, or where

   $$x + y + z = (3t + 1) + (-2t) + (t + 1) = 2t + 2 = 6.$$

This is satisfied when $t = 2$, so the intersection is at $\mathbf{r}(2) = \langle 7, -4, 3 \rangle$.

3. Find a vector equation and parametric equations for the line segment that joins $P(a, b, c)$ to $Q(u, v, w)$.

   Solution

   As in the previous problem, a vector tangent to the line is given by the vector joining the two points, or $\langle u - a, v - b, w - c \rangle$. Then using the base point P, we see that the line through points P and Q is represented as a vector function by

   $$\mathbf{r}(t) = \langle a, b, c \rangle + t \langle u - a, v - b, w - c \rangle = (1 - t) \langle a, b, c \rangle + t \langle u, v, w \rangle .$$

   Thus denoting the vector from the origin to point P as $\mathbf{P}$ and the vector from the origin to point Q as $\mathbf{Q}$, this is just $\mathbf{r}(t) = (1 - t)\mathbf{P} + t\mathbf{Q}$. In order to get just the line segment between the two points, and not the whole line, we can restrict the time parameter to the interval $[0, 1]$.

   The parametric equation is obtained from the vector equation by considering each coordinate independently.

4. Suppose that a particle's position vector is given by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. Find its position, velocity, speed, and acceleration when $t = 10$. Find the tangent line to this curve at the point $(2, 4, 8)$.

   Solution Sketch

   The general velocity and acceleration functions are obtained by differentiating coordinate-wise:

   $$\mathbf{r}'(t) = \left\langle 1, 2t, 3t^2 \right\rangle, \qquad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle .$$

   Then the speed is given by $S(t) = \|\mathbf{r}'(t)\|$, and the tangent line to the curve at the point $\mathbf{r}(s)$ is given by

   $$\mathbf{L}_s(t) = \mathbf{r}(s) + t\mathbf{r}'(s)$$

   .

5. Let $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0\mathbf{k}$.

   (a) Is $\mathbf{r}(t)$ perpendicular to $\mathbf{r}'(t)$ for every t?

   (b) Is $\mathbf{r}'(t)$ perpendicular to $\mathbf{r}''(t)$ for every t?

   (c) If $\mathbf{r}$ were another function, would the two answers above remain the same? If so, show why. If not, give a counterexample.

   Solution Sketch

   In fact, for this vector function, $\mathbf{r}(t)$ is perpendicular to $\mathbf{r}'(t)$ for every t, and also $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}''(t)$ for every t. This can be checked by taking coordinate-wise derivatives and computing the dot products of the resulting vectors, which will be zero.

   However, one cannot expect this to be the case for every vector function. For instance, consider the vector function $\mathbf{r}(t) = (e^t)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. Here we can easily see that the function and all of its derivatives are equal and non-zero, and so they are not perpendicular.