Worksheet 3 Solutions, Math 53

Vectors and Vector Products

Wednesday, September 12, 2012

1. Suppose that $\mathbf{a} \neq \mathbf{0}$.

   (a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

   (b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

   (c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

   Solution

   For the first part, the vectors need not be equal. If we rewrite the equation and factor, it is equivalent to

   $$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0,$$

   and since $\mathbf{a}$ is non-zero, this implies that $\mathbf{b} - \mathbf{c}$ is orthogonal to $\mathbf{a}$ (which includes the possibility of this vector being equal to zero).

   For the second part, similarly we find that the vectors need not be equal. In this case we have

   $$\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = 0,$$

   and so we conclude that $\mathbf{b} - \mathbf{c}$ must be parallel to $\mathbf{a}$.

   For the last part, we notice that in this case, $\mathbf{b} - \mathbf{c}$ must be both parallel and perpendicular to $\mathbf{a}$ *at the same time*, and so we conclude that it must be zero, hence $\mathbf{b}$ and $\mathbf{c}$ must be equal.

   A key point in this problem is to note the fact that two vectors having a dot product or cross product equal to zero does *not* imply that one or the other of the vectors are equal to zero as in the case of multiplying real numbers.

2. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

   Solution Idea

   Drawing a simple picture, properly labeling sides of the figure with vectors, and using basic properties of vectors is sufficient to prove this.

3. Let $r > 1$, and consider all of the points which are r times the distance from $A(-1, 0, 0)$ as from $B(1, 0, 0)$. Show that this set of points is a sphere, and find the center and radius of this sphere in terms of r.

   Solution Idea

   Let $\mathbf{X} = \langle x, y, z \rangle$, and let $\mathbf{A}$ and $\mathbf{B}$ be vector representations of points A and B respectively. Start with the vector equation

   $$\|\mathbf{X} - \mathbf{A}\| = r \, \|\mathbf{X} - \mathbf{B}\| \, ,$$

   expand it as a scalar equation, and rewrite in the standard form of a circle. The center and radius of this circle can be easily read off from this form.

4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^3$. Show that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$. What is the name of this famous theorem?

   Solution

   Writing this expression in terms of the dot product gives us

$$|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v} = |u|^2 + |v|^2 + 2(\mathbf{u}\cdot\mathbf{v})$$

   Thus we have equality exactly when $\mathbf{u}\cdot\mathbf{v} = 0$, or when the vectors are perpendicular. Drawing the vectors out head to tail shows us that this is in fact a vector formulation of the Pythagorean theorem, or more generally of the law of cosines.

5. Suppose that one side of a triangle forms the diameter of a circle and the vertex opposite this side lies on this circle. Use the dot product to prove that this is a right triangle.

   Solution Idea

   Since the geometry of the situation is preserved by scaling, translation and rotation, we can assume that the circle is the unit circle, and that the diameter stretches from $(-1, 0)$ to $(1, 0)$.

   In this case, an arbitrary point on the circle can be represented as $(\cos(\theta), \sin(\theta))$, and so we need only show that the corresponding vectors in this picture are perpendicular, which can be done by a simple calculation.