Math 480A2, Homework 9
Due October 27, 2022

Homework is graded out of a total of 10 points. Collaboration is permitted, but you must list all coauthors on a problem's solution at the top of the page, and your writing must be your own.

Problem 1. (2 points) Let ϕ be the boolean formula associated with the expression

$$\big((x_1 \wedge x_2) \vee (\bar{x}_1 \wedge \bar{x}_2)\big) \wedge (x_1 \vee x_3)$$

In the above, recall that "$\wedge$" represents an AND gate, and "$\vee$" represents an OR gate. Compute the polynomial described by the arithmetic circuit ψ associated with ϕ.

Problem 2. (3 points) Consider the following bivariate polynomials over a field K:

$$\chi_{0,0}(x,y) = (1-x)(1-y) \qquad \chi_{0,1}(x,y) = (1-x)y$$
$$\chi_{1,0}(x,y) = x(1-y) \qquad \chi_{1,1}(x,y) = xy$$

Notice that $\chi_{1,0}(1,0) = 1$, but $\chi_{1,0}(a,b) = 0$ for every other pair (a,b) of binary inputs, and that the values behave similarly for the other subscripts. Let $f : \{0,1\}^2 \to K$ be the function on binary inputs described by the following table:

x	y	$f(x,y)$
0	0	3
0	1	5
1	0	-1
1	1	1

Using the polynomials $\chi_{a,b}$ defined above, find a polynomial $\tilde{f}$ such that $\deg_x(\tilde{f}), \deg_y(\tilde{f}) \leq 1$, and such that $\tilde{f}(x,y) = f(x,y)$ for all binary inputs. The function $\tilde{f}$ is called the *multilinear extension* of f. This gives one approach for transforming an arbitrary function on binary inputs into a polynomial, for instance, for use with the sum-check protocol.

Problem 3. (1 point) Following the pattern of the functions $\chi_{a,b}$ from Problem 2, give a polynomial in $K[x_1, x_2, x_3, x_4, x_5]$ which has degree 1 in each variable, and which is equal to 1 on the input $(1,0,1,1,0)$ and 0 on every other binary input.

Problem 4. (3 points) Consider the following univariate polynomials over $\mathbb{Q}$:

$$\ell_2(x) = \frac{(x-4)(x-5)}{(2-4)(2-5)} \qquad \ell_4(x) = \frac{(x-2)(x-5)}{(4-2)(4-5)} \qquad \ell_5(x) = \frac{(x-2)(x-4)}{(5-2)(5-4)}$$

Notice that for $i \in \{2,4,5\}$, the polynomial $\ell_i(x)$ is equal to 1 when $x = i$, and is equal to 0 when x is one of the other numbers in $\{2,4,5\}$. (For instance, $\ell_2(2) = 1$, and $\ell_2(4) = \ell_2(5) = 0$.) Using these polynomials, find a polynomial $f \in \mathbb{Q}[x]$ of degree at most 2 such that $f(2) = 1$, $f(4) = -1$, and $f(5) = 2$. The polynomials ℓ_i are called the *Lagrange basis polynomials* for the base set $\{2,4,5\}$, and give a way to represent a polynomial in terms of its evaluations at a collection of points.

Problem 5. (1 point) Following the pattern of the functions ℓ_i from Problem 4, give a polynomial in $\mathbb{Q}[x]$ which has degree 3, and is equal to 1 on the point $x = 1$, and 0 on the points $x = 0, 2, 3$.