Math 480A2, Homework 7
Due October 13, 2022

Homework is graded out of a total of 10 points. Collaboration is permitted, but you must list all coauthors on a problem's solution at the top of the page, and your writing must be your own.

Problem 1. (6 points) In the following, consider the random outcomes of a fair 12-sided die, where "fair" here means that each outcome has an equal chance of occurring.

A. Describe a discrete probability space which represents the situation given above, including a set X of outcomes, and a probability function describing the likelihood of each outcome.

B. Let D be the random variable on X mapping each outcome to its number of positive divisors. (E.g. $R(6) = 4$ since 6 is divisible by 1, 2, 3, and 6.) Compute the expected value of D.

The random variable D allows us to define a probability function on the set of its possible values as follows: if d is a positive integer representing a possible number of divisors, then define $\Pr(d) = \Pr[D = d]$, where the right side of this equation is the probability of the random variable D taking value d over the original probability space X. This probability function is called the *push-forward* distribution of X by the random variable D.

C. Compute the push-forward distribution of X by the random variable D, expressed as a probability function p on the set $Y = \{1, 2, 3, \ldots\}$ of positive integers. (*Hint.* In other words, compute the values of the function $p(d) = \Pr[D = d]$ for each positive integer d.)

Problem 2. (4 points) Now consider the following random system: we repeatedly roll a fair 4-sided die until a 4 is rolled, at which point we stop, and we record the sequence of rolls as our outcome. Thus if we rolled a 3 and then a 4, the outcome would be the finite sequence $(3, 4)$, and if we rolled a 2, followed by a 1, followed by a 2, followed by a 4, then the outcome would be the sequence $(2, 1, 2, 4)$.

A. Describe a discrete probability space which represents the situation given above, including a set X of outcomes, and a probability function p describing the likelihood of each outcome. Show that $\sum_{x \in X} p(x) = 1$ for this set X and probability function p.

B. Let L be the random variable on X mapping each outcome to its number of rolls, i.e. to its length as a sequence. (E.g. $L((2, 1, 2, 4)) = 4$.) Using Markov's inequality and the fact that the expected value of L is 4, find an upper bound on the probability that a sequence of rolls has length at least 10.

Challenge. (1 bonus point) Prove that the random variable L from the previous problem has expected value 4.