Math 480A2, Homework 5

Due September 29, 2022

Homework is graded out of a total of 10 points. Collaboration is permitted, but you must list all coauthors on a problem's solution at the top of the page, and your writing must be your own.

Problem 1. (2 points) Let E be the elliptic curve over $\mathbb{R}$ defined by the short Weierstrass equation $y^2 = x^3 - x + 1$, and let $P = (1, 1)$, $Q = (-1, 1)$, and $R = (-1, -1)$ be points on E. For both of the sums $P + Q$ and $P + R$, find the line L connecting the summands, find the third intersection point of L with E, and compute the value of the sum.

Problem 2. (2 points) Let C be the curve over $\mathbb{R}$ defined by the equation $f(x, y) = xy - 1 = 0$. Give the homogenization of f, and find the points on the corresponding projective curve which lie on the line at infinity. (*Hint:* the points on the line at infinity correspond with nonzero solutions in x, y, z satisfying $z = 0$.)

Problem 3. (3 points) Let C be the curve over $\mathbb{R}$ defined by the equation $y = x^3 - x$. For $i = 1, 2, 3$, find a line L_i which intersects C at a point with multiplicity i.

Problem 4. (3 points) Let C be the curve over $\mathbb{R}$ defined by the equation $y = x^3$. The conclusion of Bézout's theorem states that a line (a curve defined by a polynomial of degree 1) should have three points of intersection with C (a curve defined by a polynomial of degree 3), with several subtleties and caveats. Each of the lines $y = 0$, $y = 1$, and $x = 0$ intersect C at only a single point in $\mathbb{R}^2$. For each, explain the sense in which it intersects C three times.