Worksheet 9 Solutions, Math 1B
Taylor and Maclaurin Series

Monday, March 12, 2012

1. Find the Taylor series for $f(x) = x^4 - 3x^2 + 1$, centered at $a = 1$ and $a = 0$.

   Solution
   We have derivatives:

Function	At x	At 1	At 0
f	$x^4 - 3x^2 + 1$	-1	1
f'	$4x^3 - 6x$	-2	0
f''	$12x^2 - 6$	6	-6
$f^{(3)}$	$24x$	24	0
$f^{(4)}$	24	24	24
$f^{(n)}$ $(n \geq 5)$	0	0	0

   Then the Taylor series centered at $a = 1$ is given by

$$\frac{-1}{0!} + \frac{-2(x-1)}{1!} + \frac{6(x-1)^2}{2!} + \frac{24(x-1)^3}{3!} + \frac{24(x-1)^4}{4!}$$
$$= -1 - 2(x-1) + 3(x-1)^2 + 6(x-1)^3 + (x-1)^4.$$

   Likewise, the Taylor series centered at $a = 0$ is given by

$$\frac{1}{0!} + \frac{-6x^2}{2!} + \frac{24x^4}{4!} = 1 - 3x^2 + x^4.$$

   Notice that the Taylor series centered at $a = 0$ is just the original polynomial. If you multiply out the Taylor series centered at $a = 1$, it will also be equal to the original polynomial. This is because the Taylor approximation is an approximation by successively higher degree polynomials, and so using a polynomial for f means that the approximations actually become exact once they reach the corresponding degree of f.

2. Find the Taylor series for $f(x) = e^x$, centered at $a = 3$.

   Solution
   All derivatives of e^x are just e^x, so $f^{(n)}(3) = e^3$ for each n. This gives us a Taylor series of

$$\sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n.$$

3. Find the Maclaurin series for $e^x + e^{2x}$.

   Solution
   Adding the corresponding series for the summands gives us

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1+2^n}{n!} x^n.$$

4. Find the Maclaurin series for $\cosh(x)$ by manipulating known series. Compare the series you find with that for $\cos(x)$.

   Solution

   We have that $\cosh(x) = 1/2 \cdot (e^x + e^{-x})$, so by combining the Maclaurin series for e^x and e^{-x}, we have

$$\frac{1}{2} \cdot \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2n!} x^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

   In particular, we notice that this is exactly the same as the series for $\cos(x)$, except that it is not an alternating series.

5. How many terms of the Maclaurin series for $\sin x$ do you need to add together in order to compute $\sin 3°$ correct to five decimal places?

   Solution Sketch

   The Maclaurin series for $\sin x$ is an alternating series with terms decreasing to 0, and so we can apply the alternating series estimation theorem to find an error bound less than $5 \cdot 10^{-6}$, which will give us five decimal places of accuracy upon rounding. It is necessary to convert $3°$ into radians before making the approximation: $3° \cdot (2\pi \text{ radians})/360°$.

6. Find the Maclaurin series for $\sin^{-1} x$. [Hint: Consider the Maclaurin series for $\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x$.]

   Solution Sketch

   We have that $\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = 1/\sqrt{1 - x^2}$, so using a binomial series expansion, we have that this derivative is equal to

$$\sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} x^{2n}$$

   for $|x| < 1$. Since this is the Maclaurin series for the derivative of $\sin^{-1}$, we can find the corresponding Maclaurin series for $\sin^{-1}$ by integrating, which gives us the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{-1/2}{n}}{2n+1} x^{2n+1} + C.$$

   Solving for C by plugging in $x = 0$ and comparing with $\sin^{-1}(0)$ gives us $C = 0$, and so we have that

$$\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{-1/2}{n}}{2n+1} x^{2n+1}$$

   for $|x| < 1$.