Worksheet 8 Solutions, Math 1B
Series Representations of Functions

Friday, March 9, 2012

2. Uses series to evaluate the limit
$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}$$

Solution Sketch

We notice that the Maclaurin series for $\cos x$ begins with constant term equal to 1, and so the difference in the numerator actually cancels out the constant term. From there we can factor out an x^2 factor:

$$\begin{aligned} 1-\cos x &= 1-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}x^{2n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{(2(n+1))!}x^{2(n+1)} = x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)!}x^{2n} \end{aligned}$$

Likewise we get similar cancellation in the denominator:

$$\begin{aligned} 1+x-e^x &= 1+x-\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=2}^{\infty}-\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}-\frac{x^{n+2}}{(n+2)!} = x^2\sum_{n=0}^{\infty}-\frac{x^n}{(n+2)!} \end{aligned}$$

Thus the overall fraction is given by

$$x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)!}x^{2n} \Bigg/ x^2\sum_{n=0}^{\infty}-\frac{1}{(n+2)!}x^n,$$

and in particular, the x^2 terms cancel. Thus we can evaluate the limit by substituting in $x=0$, which amounts to taking only the constant terms from the respective series. This gives us a limit of $(1/2)/(-1/2) = -1$.

3. Show that the function defined by
$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is not equal to its Maclaurin series.

Solution Idea

Use induction to show that each derivative of e^{-1/x^2} is of the form
$$\frac{p(x)e^{-x^2}}{q(x)}$$
where p and q are polynomials, and show that the derivative of such a function exists at $x=0$ and is equal to 0. Thus the Maclaurin series has all zero coefficients, and converges for all values of x to the constant zero function, which is not equal to $f(x)$ except at $x=0$.

4. Find the sums of the following series by comparing to known series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

Solution
This is just the exponential series using $x = 3/5$, and so it has value $e^{3/5}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

Solution
This is the cosine series with $x = \pi/6$, and so it has value $\cos(\pi/6) = \sqrt{3}/2$.

(c) $1 - \ln x + \dfrac{(\ln x)^2}{2!} - \dfrac{(\ln x)^3}{3!} + \cdots$

Solution
This is the exponential series with parameter $-\ln x$, and so the resulting value is $e^{-\ln x} = 1/x$.