Worksheet 6 Solutions, Math 1B
Comparison, Ratio, and Root Tests; Alternating Series

Monday, February 27, 2012

1. Determine whether the series converges absolutely, converges conditionally, or diverges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$

Solution Idea

Converges absolutely by the ratio test. The ratio test is a natural test to use, since the terms of the series contain a constant raised to an exponent, which cancels nicely when comparing consecutive terms.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

Solution Idea

Converges absolutely by the root test. The root test is most helpful here because the nth root of the test cancels out the nth power in the terms of the series, turning the computation into a simple limit of a rational function.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Solution Sketch

Diverges by comparison test with $\sum \frac{1}{n}$. The reason that the ratio and root tests are less useful in this case are because this series is very similar to a p-series (but not a p-series since the exponent in a p-series must be constant), and the ratio and root tests are not good at determining convergence or divergence of p-series or similar, often turning up an inconclusive result.

The computation with the comparison test will require finding the limit

$$\lim_{n\to\infty} n^{1/n}.$$

This limit has value 1, as you should verify by rewriting as

$$\lim_{n\to\infty} \exp\left(\frac{\ln(n)}{n}\right)$$

and applying L'Hôpital's rule to the expression in the exponent.

(d) $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$

Solution Idea

Converges absolutely by the ratio test.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Solution Idea

Converges by the alternating series test. Does not converge absolutely, since the comparison theorem with $\sum \frac{1}{n}$ proves divergence of $\sum |(-1)^n / \ln n| = \sum \frac{1}{\ln n}$.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

Solution Idea

Converges absolutely by the ratio test.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + 4^n}{1 + 3^n}$

Solution Idea

Diverges by the divergence test, since $(1 + 4^n)/(1 + 3^n)$ does not converge to 0 as n approaches infinity. As a rule of thumb, this is the first thing you should check when determining convergence or divergence of a series, since it is the coarsest test for divergence, and it is often very easy to check.

(h) $1 - \dfrac{1 \cdot 3}{3!} + \dfrac{1 \cdot 3 \cdot 5}{5!} - \dfrac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots + (-1)^{n-1}\dfrac{1 \cdot 3 \cdot 5 \cdots \cdots (2n-1)}{(2n-1)!} + \cdots$

Solution Idea

Converges absolutely by the ratio test. When factorials or similar products show up, very often the ratio test is the easiest way to go, since taking the ratio of consecutive terms cancels all but one term in a complicated factorial product.

2. If $\sum a_n$ is a convergent series with positive terms, is it true that $\sum \sin(a_n)$ is also convergent?

Solution Idea

This is true. Since $\sin(x) < x$ for positive x, we can use the comparison theorem with $\sum a_n$ to prove convergence.

3. If $\sum a_n$ and $\sum b_n$ are both convergent series with positive terms, is it true that $\sum a_n b_n$ is also convergent?

Solution Sketch

$\sum a_n b_n$ is also convergent. To see this, notice first that since $\sum b_n$ is convergent, $\lim_{n\to\infty} b_n = 0$, which means that for large enough n, say $n > N$ for some possibly large N, $b_n \leq 1$.

Then we can write

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{N} a_n b_n + \sum_{n=N+1}^{\infty} a_n b_n,$$

and the prior series converges exactly when the second series on the right-hand side converges. But for $n > N$, because we have that $b_n \leq 1$, we also have that $a_n b_n \leq a_n$, and so we can use the series comparison test with the convergent series $\sum_{n=N+1}^{\infty} a_n$. This completes the argument.

4. How many terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n}$$

do we need to add in order to find the sum up to an error of 10^{-4}?

Solution Idea

Check that $b_n = 1/(n5^n)$ is decreasing to zero (probably easiest to check for $f(x) = 1/(x5^x)$, which is decreasing for positive x), and find the smallest n such that $b_{n+1} \leq 10^{-4}$. This n is sufficient since the error of the nth partial sum is bounded above by b_{n+1}, by the alternating series estimation theorem.