Worksheet 5, Math 1B

Sequences and Series

Wednesday, February 22, 2012

1. Show that if $\{a_n\}$ is a sequence defined recursively by $a_{n+1} = f(a_n)$ where f is a continuous function, and if $\{a_n\}$ is convergent, then the limit of $\{a_n\}$ is a "fixed point" of f, that is, a point a such that $f(a) = a$.

2. Find an example of a recursively defined sequence $\{a_n\}$ with $a_{n+1} = f(a_n)$ for f continuous, such that f has a fixed point, and yet a_n diverges.

3. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

   satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

4. Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? In fact, a little thought reveals that this number is given by f_n, the nth Fibonacci number, defined by $f_0 = f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$.

   Let $a_n = f_{n+1}/f_n$ and show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit. What does this limit say about the behavior of the Fibonacci numbers for large values of n?

5. Determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

   (c) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2}{e^n}$

6. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1},$$

   find a_n and $\sum_{n=1}^{\infty} a_n$.

7. If $\sum a_n$ is convergent and $\sum b_n$ is divergent, show that the series $\sum (a_n + b_n)$ is divergent. [*Hint:* Argue by contradiction.]