Worksheet 5 Solutions, Math 1B

Sequences and Series

Wednesday, February 22, 2012

1. Show that if $\{a_n\}$ is a sequence defined recursively by $a_{n+1} = f(a_n)$ where f is a continuous function, and if $\{a_n\}$ is convergent, then the limit of $\{a_n\}$ is a "fixed point" of f, that is, a point a such that $f(a) = a$.

   Solution Sketch

   Let $a = \lim_{n\to\infty} a_n$. It is straightforward to show from the basic definitions that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a$. Further, we have that since f is continuous at a, $\lim_{n\to\infty} f(a_n) = f(a)$. Altogether, this gives us that
   $$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(a),$$
   which is what we wished to show.

2. Find an example of a recursively defined sequence $\{a_n\}$ with $a_{n+1} = f(a_n)$ for f continuous, such that f has a fixed point, and yet a_n diverges.

   Solution

   An example of such a sequence is the sequence with $a_0 = 1/2$, and $f(x) = 1/x$. Then the sequence alternates between $1/2$ and 2, but f has the fixed points 1 and -1.

3. Show that the sequence defined by
   $$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$
   satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

   Solution Sketch

   An argument by induction shows that $0 < a_n \leq 2$ for each n. First we have that $a_1 = 2$ satisfies the inequality. Further, if $0 < a_{n-1} \leq 2$, then $1 \leq 3 - a_n < 3$, and $0 < 1/3 < 1/(3 - a_{n-1}) = a_n \leq 1 \leq 2$, and we see that a_n satisfies the inequality as well. By induction, we can conclude that a_n satisfies the inequality for each $n \geq 1$.

   For the sequence to be decreasing, we require that $a_{n+1} \leq a_n$ for each n, and in order for this to be the case, we must have
   $$\frac{1}{3 - a_n} \leq a_n,$$
   or
   $$a_n^2 - 3a_n + 1 \leq 0.$$
   In order for this to be the case, we must have $(3 - \sqrt{5})/2 \leq a_n \leq (3 + \sqrt{5})/2$. To see that this is the case, we can modify the above argument to instead show that $(3 - \sqrt{5})/2 < a_n \leq 2$ for every n. With this bound, we have that the sequence is decreasing, and combined with the fact that the sequence is bounded below, this implies that the sequence is convergent.

By the first problem on the sheet, we then have that the limit of the sequence must be a fixed point of f, and the only fixed points of f are $(3+\sqrt{5})/2$ and $(3-\sqrt{5})/2$. The first fixed point can't be the limit, as it is larger than a_1, and the sequence is decreasing. Thus we conclude that

$$\lim_{n\to\infty} a_n = (3-\sqrt{5})/2.$$

4. Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? In fact, a little thought reveals that this number is given by f_n, the nth Fibonacci number, defined by $f_0 = f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$.

   Let $a_n = f_{n+1}/f_n$ and show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit. What does this limit say about the behavior of the Fibonacci numbers for large values of n?

   Solution

   We write
   $$a_{n-1} = \frac{f_n}{f_{n-1}} = \frac{f_{n-1}+f_{n-2}}{f_{n-1}} = 1 + \frac{f_{n-2}}{f_{n-1}} = 1 + \frac{1}{f_{n-1}/f_{n-2}} = 1 + \frac{1}{a_{n-2}},$$
   and this proves the recurrence relation.

   By the first problem on the sheet, we know that since $a_n = f(a_{n-1})$ where $f(x) = 1+1/x$, if we assume that the sequence is convergent, then the limit is a fixed point of f. Solving $f(x) = x$, we find that the fixed points of f are $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$. However, since all of the numbers a_n are positive, we can conclude that the limit must be the positive fixed point $(1+\sqrt{5})/2$, the golden ratio.

   Since the a_n are just the ratios of successive Fibonacci numbers, we see that the ratios approach a fixed number which is greater than 1. This implies that the numbers grow in a manner which is approximately exponential, with base equal to $(1+\sqrt{5})/2$, as n becomes large.

5. Determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$

   Solution Idea

   Divergent by the limit comparison test with the harmonic series.

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

   Solution Idea

   Divergent by the integral test.

   (c) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2}{e^n}$

   Solution Idea

   (Absolutely) convergent by the ratio test.

6. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1},$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

Solution

s_n is defined as the sum of the first n terms of the series, so we have

$$a_n = \left(\sum_{k=1}^{n} a_k\right) - \left(\sum_{k=1}^{n-1} a_k\right) = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{2}{n(n+1)}.$$

Then the value of the infinite sum is just the limit of the partial sums, and is given by

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$

7. If $\sum a_n$ is convergent and $\sum b_n$ is divergent, show that the series $\sum(a_n+b_n)$ is divergent. [*Hint:* Argue by contradiction.]

Solution

Suppose by way of contradiction that $\sum(a_n + b_n)$ is convergent. Then in this case, we have that since $\sum(a_n + b_n)$ and $\sum a_n$ are both convergent series, their difference is also convergent, and is given by

$$\sum(a_n + b_n) - \sum a_n = \sum(a_n + b_n - a_n) = \sum b_n.$$

But we were given that $\sum b_n$ is a divergent series, and so our conclusion that it is a convergent series is a contradiction. Our assumption that $\sum(a_n + b_n)$ was convergent must have been false, and we conclude that it must actually be divergent.