Worksheet 3 Solutions, Math 1B
Integration by Partial Fractions, Other Substitutions

Monday, January 30, 2012

1. Use the Weierstrass substitution to find the indefinite integral of $\sec(x)$. Use trigonometric identities to show that this expression is equivalent to the one derived in class.

   Solution Sketch

   The result after the substitution and integration by parts is

$$\ln\left|\frac{1+\tan(\frac{x}{2})}{1-\tan(\frac{x}{2})}\right|.$$

   Multiplying by $1+\tan(\frac{x}{2})$ in both the numerator and denominator of the trig expression and splitting gives

$$\frac{1+\tan(\frac{x}{2})}{1-\tan(\frac{x}{2})} = \frac{2\tan(\frac{x}{2})}{1-\tan^2(\frac{x}{2})} + \frac{1+\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})},$$

   and rewriting the second expression in terms if sines and cosines gives

$$\frac{\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})} = \frac{1}{\cos^2(\frac{x}{2})-\sin^2(\frac{x}{2})}.$$

   But then double angle formulas for cosine and tangent give

$$\frac{2\tan(\frac{x}{2})}{1-\tan^2(\frac{x}{2})} + \frac{1}{\cos^2(\frac{x}{2})-\sin^2(\frac{x}{2})} = \tan(x)+\sec(x),$$

   and this is the form derived in class.

2. Evaluate the following integrals:

   (a) $\displaystyle\int \frac{x^3}{x^3+1}\,dx$

   Solution Idea

   Rewrite the integral as

$$\int \frac{x^3}{x^3+1}\,dx = \int 1-\frac{1}{x^3+1}\,dx = x - \int \frac{1}{(x+1)(x^2-x+1)},$$

   and integrate by parts as usual.

   (b) $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$

   Solution Idea

   Rewrite the integral as

$$\int \frac{x^3+4}{x^2+4} = \int x - \frac{4x-4}{x^2+4}\,dx = \int x\,dx - 2\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{1+(\frac{x}{2})^2}\,dx,$$

   and each of these is solvable using previous techniques.

(c) $\displaystyle\int \frac{1}{x\sqrt{x+1}}\,dx$

Solution Sketch

This integral only makes sense for $x > -1$, so we assume that this is the case. First use a substitution such that $x+1 = u^2$, namely, $u = \sqrt{x+1}$, with $dx = 2udu$ and $x = u^2 - 1$. Then

$$\int \frac{1}{x\sqrt{x+1}}\,dx = \int \frac{2u}{(u^2-1)u}\,du = 2\int \frac{u}{(u+1)(u-1)},$$

and this we can solve as usual with integration by parts.

(d) $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$

Solution Sketch

First use a substitution such that $x^2+1 = v^3$, namely, $v = \sqrt[3]{x^2+1}$. In this case, $2x\ dx = 3v^2 dv$, and in particular, we can write

$$x^3\ dx = \frac{x^2(2x\ dx)}{2} = \frac{((x^2+1)-1)(2x\ dx)}{2} = \frac{(v^3-1)(3v^2\ dv)}{2},$$

and the integral becomes

$$\frac{3}{2}\int v(v^3-1)\ dv.$$

3. The functions $y = e^{x^2}$ and $y = x^2e^{x^2}$ don't have elementary antiderivatives, but $y = (2x^2+1)e^{x^2}$ does. Evaluate

$$\int (2x^2+1)e^{x^2}\ dx$$

Solution

We write the integral as

$$\int e^{x^2}\ dx + \int 2x^2e^{x^2},$$

and use integration by parts on the first integral, with $u = e^{x^2}$ and $dv = dx$. This gives

$$\int e^{x^2}\ dx = xe^{x^2} - \int 2x^2e^{x^2},$$

and as a consequence, we see that the overall integral has value xe^{x^2}.

4. Factor x^4+1 as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate

$$\int \frac{1}{x^4+1}\,dx$$

Solution Idea

The suggested method is a variant on completing the square, where instead of adding and subtracting a constant value, we add and subtract an intermediate value:

$$x^4+1 = x^4+2x^2+1-2x^2 = \left(x^2+1\right)^2 - \left(\sqrt{2}x\right)^2 = \left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right).$$

The quadratic formula reveals that both of these polynomials are irreducible, and so we can continue by using partial fractions with this factorization.