Worksheet 1 Solutions, Math 1B
Integration by Parts, and Trigonometric Integrals

Friday, January 20, 2012

1. Evaluate the following integrals:

   (a) $\int \cos x \, \ln(\sin x)\, dx$

   Solution Idea
   Integration by parts, using $u = \ln(\sin(x))$ and $dv = \cos(x)dx$.

   (b) $\int \sin(\ln x)\, dx$

   Solution Idea
   Integration by parts, twice. You'll end up with the integral on both sides of the equality, and rearranging using algebra allows you to solve for the value.

   (c) $\int_{\pi/6}^{\pi/3} \csc^3 x \, dx$

   Solution Sketch
   Cosecant and cotangent have a similar relationship in integration as secant and tangent. Namely, we have the identities:

   - $\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x$
   - $\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$
   - $\csc^2 x = 1 + \cot^2 x$

   Thus we may treat this integral in a similar fashion as one which is in terms of secants and tangents. In particular, we can model the solution of this integral on Example 8 from section 7.2 in Stewart, which finds the value of $\int \sec^3 x \, dx$. Thus we begin by using integration by parts, with $u = \csc x$, and $dv = \csc^2 x dx$. This yields a reduction formula, and reduces the problem to simply that of finding

   $$\int \csc x \, dx.$$

   This integral can be evaluated in a similar fashion to that of $\sec x$, in this case using $\csc x - \cot x$ as the factor to multiply and divide. This yields a value of $\ln|\csc x - \cot x| + C$ for this subintegral, and so by substituting this into the reduction formula derived above, we have a final value for the integral.

2. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

   $$\int_0^a f(x)g''(x)\, dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\, dx.$$

   Solution Idea

Integrate by parts twice, making sure to use the version of IBP corresponding to definite integrals, and using that $f(0) = g(0) = 0$ to cancel out certain terms in the resulting expression.

3. If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy.$$

Solution Sketch

First use integration by parts, and then in the remaining integral use integration by substitution with $u = f(x)$. Keep in mind the defining relation $x = g(f(x)) = g(u)$, and also recall the derivative of an inverse function, $g'(x) = 1/f'(g(x))$, which is obtained by implicit differentiation on $f(g(x)) = x$.

4. Find the volume obtained by rotating the region bounded by the curves

$$y = \sin^2 x, \quad y = 0, \quad 0 \le x \le \pi$$

about the x-axis.

Solution Idea

The standard volume construction gives the integral

$$\int_0^\pi \pi r(x)^2 dx = \pi \int_0^\pi \sin^4 x\,dx,$$

and this can be solved using the half-angle formulas.

5. Prove that for positive integers m and n,

$$\int_{-\pi}^{\pi} \sin mx\, \sin nx\,dx = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n \end{cases}.$$

Solution Idea

Use the formula $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$ from the end of section 7.2 in Stewart to convert this integral into a sum of two simpler integrals.

6. A *finite Fourier series* is given by the sum

$$f(x) = \sum_{n=1}^{N} a_n \sin nx = a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx,$$

where the coefficients a_i for $i = 1, 2, \ldots, N$ are given numbers. Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx\,dx.$$

Solution Idea

For the general m, multiply out the sum, split the integral and factor out the coefficients, and then apply the previous problem.