

Quiz 4 Solutions, Section 310

Friday, March 16, 2012

Name:

Student ID#:

Please place personal items under your seat.

No use of notes, texts, calculators, or fellow students is allowed.

Show all of your work in order to receive full credit.

1. Find the Taylor series of e^{-x} centered at $a = -1$, and determine its radius of convergence.

Solution

Every time you differentiate the function e^{-x} you get an extra factor of -1 , so if $f(x) = e^{-x}$, then

$$f^{(n)}(x) = (-1)^n e^{-x} \qquad f^{(n)}(-1) = (-1)^n \cdot e$$

and so the Taylor series of e^{-x} about $a = -1$ is given by

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot e \cdot (x+1)^n}{n!}$$

To find the radius of convergence, we use the ratio test:

$$\lim_{n \rightarrow \infty} \left| \left(\frac{(-1)^{n+1} \cdot e \cdot (x+1)^{n+1}}{(n+1)!} \right) \bigg/ \left(\frac{(-1)^n \cdot e \cdot (x+1)^n}{n!} \right) \right| = \lim_{n \rightarrow \infty} \frac{x+1}{n+1} = 0$$

So for every value of x , the limit is less than 1, and the ratio test gives us convergence. Thus the radius of convergence for this Taylor series is $+\infty$.

2. Evaluate the indefinite integral

$$\int \frac{x}{1+x^3} dx$$

using power series. Your solution should be in the form of a power series.

Solution

We expand $1/(1+x^3)$ as a power series using a geometric expansion, and simplify the integrand in this form:

$$\frac{x}{1+x^3} = x \left(\sum_{n=0}^{\infty} (-x^3)^n \right) = \sum_{n=0}^{\infty} (-1)^n x^{3n+1}$$

Integration can thus be done term by term over the power series, giving an indefinite integral of

$$\int \frac{x}{1+x^3} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{3n+2} + C$$

3. Find the degree 3 Taylor approximation of $\sin(x)$ about the point $a = \pi/4$. Find a bound for the error of this approximation at $x = 3\pi/8$ using Taylor's inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

Solution

We calculate the first, second, and third derivatives of $f(x) = \sin(x)$ at $a = \pi/4$ by

$$\begin{aligned} f'(x) &= \cos(x) & f'(\pi/4) &= \sqrt{2}/2 \\ f''(x) &= -\sin(x) & f''(\pi/4) &= -\sqrt{2}/2 \\ f'''(x) &= -\cos(x) & f'''(\pi/4) &= -\sqrt{2}/2 \end{aligned}$$

Then the degree 3 Taylor approximation of $\sin(x)$ about the point $a = \pi/4$ is given by

$$P_3(x) = \sum_{n=0}^3 \frac{f^{(n)}(a)(x-a)^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}(x-\pi/4)}{2} - \frac{\sqrt{2}(x-\pi/4)^2}{4} - \frac{\sqrt{2}(x-\pi/4)^3}{12}.$$

The variable M in Taylor's inequality is a bound on the magnitude of the fourth derivative $f^{(4)}(x) = \sin(x)$ for $|x-a| < |\pi/4 - 3\pi/8| = \pi/8$. To find this bound, we notice that $f^{(4)}(x)$ is positive and increasing on the interval $[\pi/8, 3\pi/8]$, and therefore its magnitude in this interval is bounded by its value at the right endpoint, which is $\sin(3\pi/8)$.

Thus an error bound for the approximation is given by

$$|R_3(3\pi/8)| \leq \frac{\sin(3\pi/8)}{4!} |3\pi/8 - \pi/4|^4 = \frac{\sin(3\pi/8)(\pi/8)^4}{4!}$$