Quiz 2 Solutions, Math 1B, Section 310

Friday, February 10, 2012

Name:

Student ID#:

Please place personal items under your seat.
No use of notes, texts, calculators, or fellow students is allowed.
Show all of your work in order to receive full credit.

1. Evaluate the following integral:

$$\int \frac{x-1}{x(x^2+1)}\,dx$$

Solution

We use partial fractions to simplify the integral. The partial fractions expansion is of the form

$$\frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)}$$

Equating coefficients of respective powers of x and solving the resulting system of equations gives values $A=-1$, $B=1$, and $C=1$, so

$$\begin{aligned}\int \frac{x-1}{x(x^2+1)}\,dx &= \int \frac{-1}{x} + \frac{x+1}{x^2+1}\,dx \\ &= -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\ &= -\ln|x| + \frac{1}{2}\ln(x^2+1) + \arctan(x) + C\end{aligned}$$

2. Write out, but do NOT simplify, the midpoint approximation M_4 of

$$\int_0^2 e^{-x^2/2}\,dx.$$

Calculate an error bound for this approximation using the Midpoint Rule error bound equation

$$|E_M| \le K(b-a)^3/24n^2.$$

Solution

If we write $f(x) = e^{-x^2/2}$, then the midpoint approximation M_4 with $n = 4$ is given by

$$\frac{2-0}{4}\Big(f(1/4) + f(3/4) + f(5/4) + f(7/4)\Big).$$

To calculate the error bound, we need to find an upper bound on the second derivative of f. We calculate

$$f'(x) = -xe^{-x^2/2},$$

and

$$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = (x^2-1)e^{-x^2/2} = (x+1)(x-1)e^{-x^2/2},$$

and we estimate

$$\left|f''(x)\right| = \left|(x+1)(x-1)e^{-x^2/2}\right| = |x+1|\,|x-1|\left|e^{-x^2/2}\right| \le 3\cdot 1\cdot 1 = 3.$$

Thus we can use $K = 3$ as a bound on the second derivative, and this gives us an error estimate of

$$|E_M| \le 3(2-0)^3/(24\cdot 4^2) = 1/16.$$

3. Use the comparison theorem to determine whether the following improper integral is convergent or divergent:

$$\int_1^\infty \frac{(1+\cos x)\sqrt{x-1}}{x^2}\,dx$$

Solution

We use the comparison theorem to show that the integral is convergent. Because $\cos(x)$ takes only values between -1 and 1, $0 \le (1+\cos(x)) \le 2$ for all x. Likewise, $\sqrt{x-1} \le \sqrt{x}$, so together this implies

$$0 \le \frac{(1+\cos x)\sqrt{x-1}}{x^2} \le \frac{2\sqrt{x}}{x^2} = 2x^{-3/2}.$$

But we know that $\int_1^\infty x^p\,dx$ is convergent for $p < -1$, so this means that $\int_1^\infty 2x^{-3/2}\,dx$ is convergent, and by the comparison theorem, so is $\int_1^\infty \frac{(1+\cos x)\sqrt{x-1}}{x^2}\,dx$.