Quiz 1 Solutions, Math 1B, Section 309

Friday, January 27, 2012

Name:

Student ID#:

Please place personal items under your seat.
No use of notes, texts, calculators, or fellow students is allowed.
Show all of your work in order to receive full credit.

Evaluate the following integrals (5 pts each):

1. $\displaystyle\int_1^4 x^2 \ln(x)\, dx$

   Solution

   We apply integration by parts using $u = \ln(x)$ and $dv = x^2 dx$, which gives us $du = 1/x\, dx$ and $v = x^3/3$, and

$$\begin{aligned}\int_1^4 x^2 \ln(x)\, dx = x^3 \ln(x)/3 \Big|_1^4 - \int_1^4 x^2/3\, dx &= (x^3 \ln(x)/3 - x^3/9) \Big|_1^4 \\ &= (64\ln(4)/3 - 64/9) - (1\ln(1)/3 - 1/9) = 64\ln(4)/3 - 7\end{aligned}$$

2. $\displaystyle\int \sin^2(x)\cos^2(x)\, dx$

   Solution

   Here we need to make use of the half-angle formulas, given by

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2},$$

   to reduce this to an integral in terms of only constant and linear trigonometric factors:

$$\begin{aligned}\sin^2(x)\cos^2(x) &= \frac{(1-\cos 2x)(1+\cos 2x)}{4} \\ &= \frac{1-\cos^2 2x}{4} = \frac{2-(1+\cos 4x)}{8} = \frac{1}{8}(1-\cos 4x)\end{aligned}$$

   Thus the integral is given by

$$\int \frac{1-\cos 4x}{8}\, dx = \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

3. $\displaystyle\int \frac{x^3}{\sqrt{a^2+x^2}}\,dx$

Solution

Since only a^2 appears in the integral, we can assume that $a \geq 0$ without loss of generality. Making an initial substitution of $u = x/a$ gives $du = dx/a$ and simplifies the integral to the form

$$a^3 \int \frac{u^3}{\sqrt{1+u^2}}\,du.$$

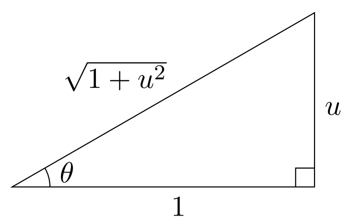
This calls for a tangent-type trig substitution, so setting $u = \tan\theta$ (where θ is chosen between $-\pi/2$ and $\pi/2$), we have

$$a^3 \int \frac{\tan^3\theta}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta = a^3\int \tan^3\theta\sec\theta\,d\theta = a^3\int \sec^3\theta\tan\theta\,d\theta - a^3\int \sec\theta\tan\theta\,d\theta$$

The equalities above both come from using the Pythagorean identity $1+\tan^2\theta = \sec^2\theta$. Using the substitution $v = \sec\theta$ in each case, we find a final value for the integrals to be

$$a^3\sec^3\theta/3 - a^3\sec\theta + C,$$

and we finish by substituting back. In particular, $\sec\theta = \sec(\tan^{-1}u)$, and so by drawing a right triangle:



we see that $\sec\theta = \sqrt{1+u^2} = \sqrt{a^2+x^2}/a$, and so the integral evaluates to

$$(a^2+x^2)^{3/2}/3 - a^2(a^2+x^2)^{1/2} + C = \sqrt{a^2+x^2}\left(x^2-2a^2\right)/3 + C.$$