Name:

Math 110, Section 101, Quiz 13

Wednesday, November 29, 2017

This quiz will be graded out of 15 points; the True/False question is worth 3 points, and the exercise is worth 12 points. Please read the instructions carefully, and explain your work.

True or False. Mark the following statements as either true or false, or leave a blank if you don't know. A correct answer is worth +1 point, a blank is worth 0 points, and an incorrect answer is worth -1 points, so be smart about guessing!

a. ________ If $T : \mathbb{C}^n \to \mathbb{C}^n$ is a linear transformation and β is any ordered basis of $\mathbb{C}^n$, then the matrix $[T^*]_\beta$ of the adjoint operator T^* is given by $([T]_\beta)^*$.

b. ________ Adjoints exist for every linear operator on a finite-dimensional inner product space.

c. ________ Any self-adjoint operator in a finite-dimensional inner product space is normal.

Solution. F T T

✷

Exercise. In $P_2(\mathbb{R})$ equipped with the inner product $\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)\,dt$, find the adjoint of the derivative operator $T : p \mapsto p'$. Represent the resulting operator in terms of polynomials.

Hint: What is the matrix of T using the orthonormal basis $\beta = \{\sqrt{1/2}, \sqrt{3/2}x, \sqrt{5/8}(3x^2 - 1)\}$?

Solution. The derivative operator on the vectors in β is given by

$$\begin{aligned} T(\sqrt{1/2}) &= 0 \\ T(\sqrt{3/2}x) &= \sqrt{3/2} = \sqrt{3}\sqrt{1/2} \\ T(\sqrt{5/8}(3x^2-1)) &= 6\sqrt{5/8}x = (3\sqrt{5/3})\sqrt{3/2}x \end{aligned}$$

Thus the matrix of T is given by

$$[T]_\beta = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & 3\sqrt{5/3} \\ 0 & 0 & 0 \end{pmatrix}$$

Since β is orthonormal, we have that $[T^*]_\beta = [T]_\beta^*$. Thus

$$[T^*]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 3\sqrt{5/3} & 0 \end{pmatrix}$$

In particular, T^* may be represented for a polynomial as

$$T^*\left(a\sqrt{1/2} + b\sqrt{3/2}x + c\sqrt{5/8}(3x^2-1)\right) = \left(\sqrt{3}a\right)\sqrt{3/2}x + \left(3\sqrt{5/3}b\right)\sqrt{5/8}(3x^2-1)$$