Name:

Math 110, Section 101, Quiz 9
Wednesday, October 25, 2017

This quiz will be graded out of 15 points; the True/False question is worth 3 points, and the exercise is worth 12 points. Please read the instructions carefully, and explain your work.

True or False. Mark the following statements as either true or false, or leave a blank if you don't know. A correct answer is worth +1 point, a blank is worth 0 points, and an incorrect answer is worth -1 points, so be smart about guessing!

a. ________ If T is a linear operator on a vector space V, then any T-cyclic subspace of V is T-invariant.

b. ________ If T is a diagonalizable linear operator, then every T-cyclic subspace has dimension 1.

c. ________ Any eigenspace of a linear operator T is T-invariant.

Solution. T F T

Exercise. Determine whether the following linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ is diagonalizable, but don't diagonalize:

$$T(x, y, z) = (7x - 4y, 8x - 5y, 6x - 6y + 3z)$$

Explain your steps carefully.

Solution. T is diagonalizable iff its characteristic polynomial splits over $\mathbb{R}$, and the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity. We compute the characteristic polynomial of T with a determinant,

$$\begin{aligned}\det\begin{pmatrix} 7-\lambda & -4 & 0 \\ 8 & -5-\lambda & 0 \\ 6 & -6 & 3-\lambda \end{pmatrix} &= (7-\lambda)(-5-\lambda)(3-\lambda) - (8)(-4)(3-\lambda) \\ &= -\big((\lambda-7)(\lambda+5)+32\big)(\lambda-3) \\ &= -(\lambda-3)^2(\lambda+1)\end{aligned}$$

Thus the characteristic polynomial splits over $\mathbb{R}$. Since the geometric multiplicity of an eigenvalue is always at least one, the algebraic and geometric multiplicities of $\lambda = -1$ are both equal to 1. For $\lambda = 3$, we need to check that the null space of $T - \lambda I$ has dimension 2. Row reducing, we have

$$\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $T - \lambda I$ has rank 1 for $\lambda = 3$, and so nullity 2. This means $\lambda = 3$ has geometric multiplicity 2. We conclude that T is diagonalizable.