Math 110, Section 101, Quiz 5
Wednesday, September 27, 2017

This quiz will be graded out of 15 points; the True/False question is worth 3 points, and the exercise is worth 12 points. Please read the instructions carefully, and explain your work.

True or False. Mark the following statements as either true or false, or leave a blank if you don't know. A correct answer is worth +1 point, a blank is worth 0 points, and an incorrect answer is worth -1 points, so be smart about guessing!

a. ________ If β and β' are finite bases of a vector space V, and if $T : V \to V$ is a linear transformation, then the change of basis matrix Q which satisfies $[T]_\beta = Q^{-1}[T]_{\beta'}Q$ is given by $[I_V]_\beta^{\beta'}$.

b. ________ Two matrices A and B in $M_{n\times n}(F)$ are called similar if $AB - BA = 0$, and this is an equivalence relation on $M_{n\times n}(F)$.

c. ________ Any invertible $n\times n$ matrix is a change of basis matrix for some linear transformation with respect to some pair of ordered bases.

Solution. T F T

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Exercise. If $\beta = \{(1,2),(2,-1)\}$ and $\gamma = \{(4,3),(5,0)\}$ are two ordered bases of $\mathbb{R}^2$, find the change of coordinate matrix that changes γ-coordinates into β-coordinates.

Solution. By inspection, since the vectors are small enough to work with easily, we can see that

$$(4,3) = 2\cdot(1,2) + 1\cdot(2,-1), \quad (5,0) = 1\cdot(1,2) + 2\cdot(2,-1)$$

Thus we can use the coefficients of these linear combinations as the columns of the matrix in question, to obtain the change of coordinate matrix

$$Q = [\mathrm{Id}]_\gamma^\beta = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Alternatively, we can use the above vectors to form the matrices

$$[\mathrm{Id}]_\gamma^E = \begin{pmatrix} 4 & 5 \\ 3 & 0 \end{pmatrix}, \quad [\mathrm{Id}]_\beta^E = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

where E is the standard basis of $\mathbb{R}^2$. Thus

$$[\mathrm{Id}]_E^\beta = ([\mathrm{Id}]_\beta^E)^{-1} = \frac{1}{1\cdot -1 - 2\cdot 2}\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}$$

so

$$[\mathrm{Id}]_\gamma^\beta = [\mathrm{Id}]_E^\beta[\mathrm{Id}]_\gamma^E = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$