Math 110, Section 105, Quiz 3
Wednesday, September 13, 2017

This quiz will be graded out of 15 points; the True/False question is worth 3 points, and the exercise is worth 12 points. Please read the instructions carefully, and explain your work.

True or False. Mark the following statements as either true or false, or leave a blank if you don't know. A correct answer is worth +1 point, a blank is worth 0 points, and an incorrect answer is worth -1 points, so be smart about guessing!

a. ________ In $M_{2\times 2}(\mathbb{R})$, the function $T : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b & c+d \\ a-c & b-d \end{pmatrix}$ is linear.

b. ________ If V is a vector space over $\mathbb{R}$, and $x, y, z \in V$ form a basis, then $x+y$, $y+z$, and $x+z$ also form a basis of V.

c. ________ If V and W are vector spaces and $T : V \to W$ is a one-to-one linear function, then the image of any linearly independent set in V is linearly independent in W.

Solution. T T T

✷

Exercise. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map for which we know the values on a basis:

$$T(\,(1,0,0)\,) = (1,1,0)$$
$$T(\,(1,1,0)\,) = (0,1,1)$$
$$T(\,(1,1,1)\,) = (1,0,1)$$

For a general point $(x, y, z) \in \mathbb{R}^3$, compute $T((x, y, z))$.

Solution. We use the fact that any vector (x, y, z) can be represented uniquely as a linear combination of basis vectors to evaluate the linear transformation. Namely, we can see that

$$\begin{aligned}
(x,0,0) &= x \cdot (1,0,0) \\
(0,y,0) &= -y \cdot (1,0,0) + y \cdot (1,1,0) \\
(0,0,z) &= - z \cdot (1,1,0) + z \cdot (1,1,1)
\end{aligned}$$

In particular, this means that

$$(x, y, z) = (x-y) \cdot (1,0,0) + (y-z) \cdot (1,1,0) + z \cdot (1,1,1)$$

Thus using linearity of T, we can write

$$\begin{aligned}
T((x,y,z)) &= (x-y)T((1,0,0)) + (y-z)T((1,1,0)) + zT((1,1,1)) \\
&= (x-y)(1,1,0) + (y-z)(0,1,1) + z(1,0,1) \\
&= (x-y+z, x-z, y)
\end{aligned}$$